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Optimization and asymptotics of eigenvalues for differential operators with surface interactions

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Abstract

The present thesis is devoted to geometric aspects of spectral theory of differential operators with surface interactions. We consider three types of operators with surface interactions: the Robin Laplacian, the Dirac operator with the infinite mass boundary condition, and the Schrödinger operator with a singular interaction supported on a surface. For the Robin Laplacian we obtain spectral isoperimetric inequalities for low eigenvalues in various settings. Furthermore, we derive geometric bounds on the principal eigenvalue of the Dirac operator with the infinite mass boundary condition. Finally, for the Schrödinger operator with a singular interaction we prove spectral isoperimetric inequalities for the lowest eigenvalue and in several settings obtain spectral asymptotics, in which the shape of the interaction support manifests.

Contents

1 Introduction

The study of connections between spectral theory and geometry constitutes an important area of mathematics called *spectral geometry*. Such connections are of interest from a purely mathematical point of view and also due to their applications in various branches of physics, since these connections yield new information about properties of differential equations describing physical systems.

In the present thesis, we discuss shape optimization of the eigenvalues and manifestation of the geometry in their asymptotic behaviour for differential operators describing physical systems with surface interactions. We mainly consider three types of differential operators with surface interactions: the *Robin Laplacian*, the *Dirac operator with the infinite mass boundary condition*, and the *Schrödinger operator with a singular interaction* supported on a surface. Spectral problems for these operators have elegant formulations and are of interest in pure mathematics. They also find applications in the description of physical systems. The investigation of the spectra of the Robin Laplacian is partially motivated by applications in elasticity theory and superconductivity. The Dirac operator with the infinite mass boundary condition appears in the description of graphene quantum dots. Finally, singular interactions supported on a surface serve as idealized models of regular potentials localized in the vicinity of this surface, which often arise in mesoscopic physics. Further, we briefly outline the obtained results and provide the historical context. More detailed discussion of the results is presented in Sections 2-4 below.

The Robin Laplacian

In our analysis of the Robin Laplacian we focus on optimization of the low lying Robin eigenvalues with respect to the shape of the domain. We prove spectral isoperimetric inequalities for Robin eigenvalues with negative boundary parameter in several related settings. In the papers [KL18, KL20] we obtain results on optimization of the lowest Robin eigenvalue in the exterior of a bounded set in all space dimensions. In the exterior of a convex planar set, we prove in [EL22] a spectral isoperimetric inequality for the second eigenvalue. We also obtain results on the optimization of the lowest Robin eigenvalue in unbounded three-dimensional cones and on two-dimensional simply-connected manifolds [KhL22]. Furthermore, we address in [KaL22] optimization of the lowest magnetic Robin eigenvalue in two dimensions for the moderate intensity of the homogeneous magnetic field. Besides the Robin Laplacian we also deal with closely related Robin bi-Laplacian. We consider in [L23] optimization of the lowest eigenvalue of perturbed Robin bi-Laplacian in a planar exterior domain. All the analysed settings have not been considered in the literature before in light of eigenvalue optimization.

The study of spectral isoperimetric inequalities goes back to the monograph *The Theory of Sound* by Lord Rayleigh [R]. First rigorous results for the lowest eigenvalue of the Dirichlet Laplacian are proved a century ago by Faber [F23] and Krahn [K24]. Spectral isoperimetric inequalities for the lowest Robin eigenvalue are obtained much later: for the positive boundary parameter by Bossel [B86] and Daners [D06] and for the negative boundary parameter by Antunes, Freitas, and Krejčiřík [AFK17] and by Bucur, Ferone, Nitsch, and Trombetti [BFNT18]. Related results are proven by many authors, some of which are reviewed in Section 2.

Dirac operator with the infinite mass boundary condition

We consider in the papers [ABLO21, LO19] the two-dimensional massless Dirac operator on a bounded domain with the infinite mass boundary condition. Our main results concern sharp upper bounds on the smallest positive eigenvalue of this operator. The upper bound is expressed through the smallest positive eigenvalue of the disk, the area of the domain, its perimeter, and the in-radius. The bound is sharp in the sense that for the disk it becomes an inequality. In order to obtain an upper bound in [ABLO21] we derived a new variational principle of independent interest for the Dirac operator, which transforms the two-component spectral problem for this operator into one-component spectral problem for the Laplacian with oblique derivative boundary condition and the spectral parameter enters also in the boundary condition. Moreover, we provide a numerical evidence, which shows that the smallest positive eigenvalue of this Dirac operator is minimized by the disk among all simply-connected domains of fixed area. Further details on these results are provided in Section 3.

The two-dimensional Dirac operator with the infinite mass boundary condition was rigorously introduced by Benguria, Fournais, Stockmeyer, and Van Den Bosch in [BFSV17a] in connection with graphene quantum dots. The same group of authors obtained in [BFSV17b] a lower bound on the smallest positive eigenvalue of this operator in terms of the area of the domain. Operators of this kind were also investigated earlier on manifolds with a purely geometric motivation; see *e.g.* [HMZ01, R06].

Schrödinger operator with a singular interaction

In the series of papers [BEL14, EL17, LO16] we analysed the Schrödinger operator with an attractive δ-interaction supported on an unbounded conical surface. In [BEL14] we proved that the discrete spectrum below the threshold of the essential spectrum induced by the circular conical surface in three dimensions is infinite. We computed the respective spectral asymptotics in [LO16] and showed that the eigenvalues are non-decreasing functions of the aperture of the conical surface. Moreover, we proved in [LO16] that in space dimensions $d \geq 4$ the circular conical surface does not induce bound states and the spectrum is purely continuous. In [EL17] we considered optimization of the lowest eigenvalue induced by the conical surface in three dimensions of generic cross-section. Our analysis was partially inspired by the results on the discrete spectrum for the Dirichlet Laplacian on a conical layer by Duclos, Exner, and Krejčiřík [DEK01], Exner and Tater [ET10] and Dauge, Ourmières-Bonafos, and Raymond [DOR15].

In [EKL18] we analysed the discrete spectrum induced by the attractive δ-interaction supported on a locally and weakly deformed plane in three dimensions. We proved that for a sufficiently weak deformation there is a unique simple eigenvalue below the threshold of the essential spectrum and obtained its asymptotics in terms of the profile of the deformation. This analysis was motivated by partial results [EK03] due to Exner and Kondej on the existence of bound states below the threshold of the essential spectrum induced by an attractive δ-interaction supported on an asymptotically flat surface in three dimensions. It remains an open problem whether any such surface induces a non-empty discrete spectrum for any strength of the attractive δ-interaction. Our results in [EKL18] give some insights on this open problem.

In the series of papers [EL21, L19, L21] we analysed optimization of the lowest eigenvalue induced by attractive (singular) interactions in two dimensions in several related settings. In particular, in [L19] we considered optimization of the lowest eigenvalue induced by δ-interaction supported on an open arc of fixed length with two endpoints. Further, in [L21] we obtained an optimization result for the lowest eigenvalue induced by an attractive δ'-interaction supported on a contour of fixed length. Finally, in [EL21] we proved isoperimetric inequalities for the lowest eigenvalue in the model of soft quantum ring, namely, a regular potential supported in the curved strip in the plane. These considerations are inspired by the result in [EHL06] due to Exner, Harrell, and Loss on the optimization of the lowest eigenvalue induced by the δ-interaction supported on a contour in two dimensions.

In the paper [BEHL20] we considered the Landau Hamiltonian perturbed by the δ-interaction supported on a curve. Our main result in this setting concerns the asymptotics of the accumulation of the eigenvalues at the Landau levels. This spectral asymptotics is expressed in terms of the intensity of the magnetic field and the logarithmic capacity of the interaction support. This result is a counterpart for δ-interactions of the result by Raikov [R90] on regular potentials and by Pushnitski and Rozenblum [PR07] on the magnetic Dirichlet Laplacian on an exterior domain.

2 Optimization of the Robin eigenvalues

The Robin eigenvalue problem for the Laplace operator is considered in a vast number of publications during the last century. Besides purely mathematical interest explained by elegance of this problem, there is a classical application of the Robin Laplacian in physics to the elasticity theory in the description of the elastically supported membrane. Another application the Robin Laplacian finds in the theory of superconductivity [GS07].

For a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with sufficiently smooth boundary $\partial\Omega$, we consider the spectral problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \partial_\nu u + \beta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\partial_\nu u$ denotes the normal derivative of u with the normal pointing outwards of Ω and $\beta \in \mathbb{R}$ is called the boundary parameter. The problem (2.1) can be interpreted as a spectral problem for a rigorously defined semibounded self-adjoint operator

$$\begin{aligned} \mathsf{H}_\beta^\Omega u &:= -\Delta u, \\ \operatorname{dom} \mathsf{H}_\beta^\Omega &:= \big\{ u \in H^1(\Omega) \colon \Delta u \in L^2(\Omega),\ \partial_\nu u + \beta u = 0 \,\text{on}\, \partial\Omega \big\}, \end{aligned} \tag{2.2}$$

acting in the Hilbert space $L^2(\Omega)$. The spectrum of H_β^Ω is purely discrete and we denote by $\{\lambda_k^\beta(\Omega)\}_{k\geq 1}$ the eigenvalues of this operator enumerated in the non-decreasing order and repeated with multiplicities taken into account. The sign of $\lambda_1^\beta(\Omega)$ is the same as of the parameter β and $\lambda_1^0(\Omega) = 0$. We also include the case $\beta = \infty$ into the discussion which formally corresponds to the Dirichlet boundary condition.

The case $\beta = \infty$ (Dirichlet). In the case of the Dirichlet boundary condition it is proved by Faber and Krahn a century ago that the ball is the minimizer of the lowest eigenvalue among domains of the same volume [F23, K24]. This

statement was conjectured Lord Rayleigh in his famous monograph *The Theory of Sound* [R]. The second Dirichlet eigenvalue is minimized by the union of two disjoint identical balls among domains of the same volume [K26].
The case $\beta > 0$. It was proved by Bossel [B86] in two dimensions and then by Daners [D06] in higher dimensions that for $\beta > 0$ the eigenvalue $\lambda_1^\beta(\Omega)$ is minimized by the ball among all domains of the same volume. The second eigenvalue $\lambda_2^\beta(\Omega)$ is shown in [K09] to be minimized under fixed volume constraint by the union of two disjoint balls of the same radius. In view of these results the understanding of the optimization for the lowest two eigenvalues for $\beta > 0$ can be regarded as quite complete. The techniques of the proofs are significantly different from the one used in the case of the Dirichlet boundary condition.
The case $\beta < 0$. It was conjectured by Bareket [B77] that for $\beta < 0$ under the fixed volume constraint the ball is the maximizer of the lowest Robin eigenvalue based on the local optimality of the ball, but it was later shown by Freitas and Krejčiřík [FK15] that it is not true, because the spherical shell of the same volume as the ball gives larger lowest eigenvalue for $\beta < 0$ with sufficiently large absolute value. However, under the fixed perimeter constraint the disk is proved by Antunes, Freitas, and Krejčiřík [AFK17] to be the maximizer of $\lambda_1^\beta(\Omega)$. It is also conjectured [AFK17, Conj. 2] that under the fixed area constraint the disk is the maximizer of $\lambda_1^\beta(\Omega)$ in the class of simply connected domains. In higher space dimensions the ball is proved in [BFNT18] to be the maximizer of $\lambda_1^\beta(\Omega)$ under fixed area of the boundary in the class of convex domains and it is conjectured in [AFK17, Conj. 4] that the convexity assumption can be dropped. The results on the optimization for the second eigenvalue $\lambda_2^\beta(\Omega)$ for $\beta < 0$ are less complete. It is proved in [FL20, FL21] that $\lambda_2^\beta(\Omega)$ is maximized by the ball under fixed volume constraint provided that $\beta \in [-\frac{d+1}{d}R^{-1}, 0]$, where R is the radius of the optimal ball.

We address the optimization of the lowest Robin eigenvalue and in some cases of the second Robin eigenvalue for the negative boundary parameter in several modified settings. First, we consider the optimization of the low Robin eigenvalues in the complement of a bounded open set. Second, we study the optimization of the lowest Robin eigenvalue on two-dimensional simply-connected Riemannian manifolds with boundary and in three-dimensional unbounded Euclidean cones. Finally, we modify the optimization problem on a bounded Euclidean domain by adding a homogeneous magnetic field.

Besides the Laplace operator we also deal with the bi-Laplacian. In general, the eigenvalue optimization for the bi-Laplacian is more complicated than for the Laplacian. Nadirashvili proved in [N95] the analogue of the Faber-Krahn inequality for the Dirichlet bi-Laplacian (describing the clamped plate) in dimension $d = 2$. Ashbaugh and Benguria [AB95] obtained such an isoperimetric inequality in dimension $d = 3$. The same question in dimensions $d \geq 4$ remains still open. An analogue of the Szegő-Weinberger inequality for the perturbed Neumann bi-Laplacian corresponding to a plate under tension with free boundary was proved by Chasman in [C11]. Here, the considered perturbation corresponding to tension is of lower order. The perturbed Robin bi-Laplacian in a bounded domain has been recently introduced by Chasman and Langford [CL20] and in a more general form by Buoso and Kennedy [BK22]. The spectral analysis of this operator is partially motivated by application in mechanics to the study of vibrations of plates with elastic response of the boundary. An isoperimetric inequality for the lowest eigenvalue of the perturbed Robin bi-Laplacian in a bounded domain remains an open problem.

2.1 Exterior of a compact set ([KL18, KL20, EL22, L23])

In this series of papers we considered optimization of the low Robin eigenvalues in the exterior of a compact set. Recall that Ω stands for a bounded open set in $\mathbb{R}^d$ with sufficiently smooth boundary. We point out that Ω is not assumed to be connected, but in this subsection we additionally assume that all the connected components of Ω are simply connected. We denote the complement of Ω by $\Omega^{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}$. The domain $\Omega^{\rm ext}$ is unbounded, but its boundary is compact. We consider the Robin Laplacian on the exterior domain $\Omega^{\rm ext}$

$$\begin{aligned} \mathsf{H}_\beta^{\Omega^{\rm ext}} u &:= -\Delta u, \\ \operatorname{dom} \mathsf{H}_\beta^{\Omega^{\rm ext}} &:= \big\{ u \in H^1(\Omega^{\rm ext}) \colon \Delta u \in H^1(\Omega^{\rm ext}),\, \partial_\nu u - \beta u = 0 \text{ on } \partial\Omega \big\}, \end{aligned}$$

where the change of sign in the boundary condition is related to the fact that the outer normal vector for Ω is the inner normal vector for $\Omega^{\rm ext}$. It can be easily shown that the operator $\mathsf{H}_\beta^{\Omega^{\rm ext}}$ is self-adjoint in the Hilbert space $L^2(\Omega^{\rm ext})$. One of the main differences from the bounded domain case is that the essential spectrum of $\mathsf{H}_\beta^{\Omega^{\rm ext}}$ is not empty and coincides with the interval $[0, \infty)$ for all $\beta \in \mathbb{R}$. For $\beta \geq 0$ there is no spectrum other than this essential spectrum, while for $\beta < 0$ there can be finitely many negative eigenvalues. In what follows we assume that $\beta < 0$ and denote by $\{\lambda_k^\beta(\Omega^{\rm ext})\}_{k\geq 1}$ the negative eigenvalues of $\mathsf{H}_\beta^{\Omega^{\rm ext}}$ enumerated in the non-decreasing order and counted with the multiplicities. This sequence is extended up to an infinite one repeating the value 0 infinitely many times. We are interested in the optimization of

$\lambda_1^\beta(\Omega^{\rm ext})$ and of $\lambda_2^\beta(\Omega^{\rm ext})$ with respect to the variation of the shape of Ω under various natural geometric constraints. The results are convenient to split into the two-dimensional case and the higher dimensional case.

Two dimensions ($d = 2$). In this setting there is at least one negative eigenvalue for all $\beta < 0$. In the case that Ω is connected we prove in [KL18, KL20] that $\lambda_1^\beta(\Omega^{\rm ext}) < 0$ is maximized by the exterior of the disk under both constraints $|\Omega| = \text{const}$ and $|\partial\Omega| = \text{const}$. The situation becomes more involved if Ω is not connected. We assume that Ω has $N \geq 2$ connected components and in this case we prove in [KL20] that

$$\lambda_1^\beta(\Omega^{\rm ext}) \leq \lambda_1^\beta(\mathcal{B}^{\rm ext}), \qquad \text{for all } \beta < 0,$$

where $\mathcal{B} \subset \mathbb{R}^2$ is the disk satisfying $\frac{|\partial\Omega|}{N} = |\partial\mathcal{B}|$. We also show by constructing a counterexample in [KL18] that in the last condition the number of connected components can not be removed from the denominator.

For optimization of the second Robin eigenvalue we restrict the class of two-dimensional exterior domains by complements of bounded convex planar sets. We also denote by $\kappa_{\partial\Omega} \geq 0$ the curvature of the boundary $\partial\Omega$ of a convex domain Ω. We prove in [EL22] that for a convex domain $\Omega \subset \mathbb{R}^2$ the inequality

$$\lambda_2^\beta(\Omega^{\rm ext}) \leq \lambda_2^\beta(\mathcal{B}^{\rm ext}), \qquad \text{for all } \beta < 0,$$

holds under the constraint $\max \kappa_{\partial\Omega} \leq \frac{1}{R}$, where R is the radius of the disk $\mathcal{B}$. It remains an open question whether the same inequality holds under fixed perimeter constraint. The advantage of this result on the optimization of the second Robin eigenvalue in comparison with the results for a bounded domain [FL20, FL21] is that it holds without any restrictions on the boundary parameter.

Higher dimensions ($d \geq 3$). In this setting it turns out that there exists a critical boundary parameter $\beta_\star = \beta_\star(\Omega^{\rm ext}) < 0$ such $\lambda_1^\beta(\Omega^{\rm ext}) < 0$ if, and only if $\beta < \beta_\star$. For higher space dimensions we show by a counterexample in [KL18] that under the constraint of fixed area of the boundary the exterior of the ball can not be the maximizer for $\lambda_1^\beta(\Omega^{\rm ext})$ for all $\beta < 0$ even in the class of convex Ω's. The domain Ω which plays the role of a counterexample is constructed as the convex hull of two disjoint identical balls whose centers are sufficiently far away from each other. Choosing the parameters so that the area of the boundary of this convex hull $\Omega \subset \mathbb{R}^d$ is the same as of the ball $\mathcal{B} \subset \mathbb{R}^d$ we show that $\lambda_1^\beta(\Omega^{\rm ext}) > \lambda_1^\beta(\mathcal{B}^{\rm ext})$ for all $\beta < 0$ sufficiently large by absolute value. This counterexample shows that in higher space dimensions the optimization in the exterior of compact sets is significantly different from the case of bounded domains where it is proved in [BFNT18] that fixed area of the boundary ensures optimality of the ball in the class of convex domains and moreover this convexity assumption can be potentially removed as the numerical evidence in [AFK17] shows.

The natural questions arises whether one can prove a spectral isoperimetric inequality under a possibly different geometric constraint. We found such a constraint which involves the mean curvature. For a bounded convex domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, we introduce the geometric functional

$$\mathcal{M}(\partial\Omega) := \frac{\displaystyle\int_{\partial\Omega} M^{d-1}(x)\mathsf{d}\sigma(x)}{|\partial\Omega|},$$

where $M \geq 0$ is the mean curvature of $\partial\Omega$. We prove that for a bounded convex domain $\Omega \subset \mathbb{R}^d, d \geq 3$, the following inequality

$$\lambda_1^\beta(\Omega^{\rm ext}) \leq \lambda_1^\beta(\mathcal{B}^{\rm ext}), \qquad \text{for all } \beta < 0,$$

where $B \subset \mathbb{R}^d$ is the ball satisfying $\mathcal{M}(\partial\Omega) = \mathcal{M}(\partial\mathcal{B})$. Moreover, a reverse inequality holds between the critical boundary parameters for $\Omega^{\rm ext}$ and $\mathcal{B}^{\rm ext}$ under the same geometric constraint.

Now we pass to the discussion of the results on the bi-Laplacian. As it was already mentioned, an isoperimetric inequality for the lowest eigenvalue of the perturbed Robin bi-Laplacian in a bounded domain is an open problem. Motivated by this open problem we considered an optimization of the lowest eigenvalue of this operator in a complementary setting of an exterior domain. Let the space dimension $d = 2$, the boundary parameter $\gamma < 0$, and the tension parameter $\alpha \geq 0$ be fixed. We introduce the perturbed Robin bi-Laplacian $\mathsf{H}_{\alpha,\gamma}^{\Omega^{\rm ext}}$ as the unique self-adjoint operator in $L^2(\Omega^{\rm ext})$ associated with the closed, densely defined, symmetric and lower-semibounded quadratic form

$$\begin{aligned} H^2(\Omega^{\rm ext}) \ni u \mapsto \|\nabla\partial_1 u\|^2_{L^2(\Omega^{\rm ext};\mathbb{C}^2)} + \|\nabla\partial_2 u\|^2_{L^2(\Omega^{\rm ext};\mathbb{C}^2)} \\ + \alpha\|\nabla u\|^2_{L^2(\Omega^{\rm ext};\mathbb{C}^2)} + \gamma\|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)}. \end{aligned}$$

We show that the essential spectrum of this operator coincides with the set $[0,\infty)$ and that the negative discrete spectrum is non-empty. Denote by $\lambda_1^{\alpha,\gamma}(\Omega^{\rm ext}) < 0$ the lowest eigenvalue of $\mathsf{H}_{\alpha,\gamma}^{\Omega^{\rm ext}}$. Assume, in addition, that Ω is convex and let $\kappa_{\partial\Omega} \geq 0$ be the curvature of its boundary. Let $\mathcal{B}$ be the disk of the radius $R > 0$. Under the assumption $\alpha \geq \frac{1}{R^2}$ and the constraint $\max \kappa_{\partial\Omega} \leq \frac{1}{R}$ we prove in [L23] that

$$\lambda_1^{\alpha,\gamma}(\Omega^{\rm ext}) \leq \lambda_1^{\alpha,\gamma}(\mathcal{B}^{\rm ext}),$$

where the equality is attained if, and only if, Ω and $\mathcal{B}$ are congruent.

2.2 Unbounded cones ([KhL22])

In the previous subsection we discussed the optimization of low Robin eigenvalues for a class of unbounded domains with compact boundaries. In this subsection we consider a special class of unbounded three-dimensional domains with non-compact boundaries. Let $\mathfrak{m} \subset \mathbb{S}^2$ be a simply-connected smooth open set in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. We define the unbounded three-dimensional cone with the cross-section $\mathfrak{m}$ as

$$\Lambda_{\mathfrak{m}} := \{ r\mathfrak{m} \colon r > 0 \}.$$

The Robin Laplacian $\mathsf{H}_\beta^{\Lambda_\mathfrak{m}}$ on $\Lambda_\mathfrak{m}$ is defined as in (2.2) with Ω replaced by $\Lambda_\mathfrak{m}$. In the following we assume that $\beta < 0$ in order to have a non-trivial spectral portrait. According to [P16] the essential spectrum of $\mathsf{H}_\beta^{\Lambda_\mathfrak{m}}$ coincides with the interval $[-\beta^2, \infty)$ and provided that $|\mathfrak{m}| < 2\pi$ the discrete spectrum below the point $-\beta^2$ is infinite. The asymptotics of these eigenvalues is computed in [BPP18]. One of the motivations to consider Robin Laplacians on unbounded cones stems from the fact that they appear in the asymptotic analysis of the Robin Laplacian on a bounded non-smooth domain with conical points in the regime $\beta \to -\infty$.

We denote by $\lambda_1^\beta(\Lambda_\mathfrak{m}) < -\beta^2$ the lowest eigenvalue of the Robin Laplacian on the cone $\Lambda_\mathfrak{m}$ with the boundary parameter $\beta < 0$. In [KhL22] we proved that

$$\lambda_1^\beta(\Lambda_\mathfrak{m}) \le \lambda_1^\beta(\Lambda_\mathfrak{b}) = -\frac{4\pi^2\beta^2}{|\partial\mathfrak{b}|^2}, \qquad \text{for all}\, \beta < 0,$$

where $\mathfrak{b} \subset \mathbb{S}^2$ is the spherical cap of the same perimeter as $\mathfrak{m}$ and provided that $|\mathfrak{m}|, |\mathfrak{b}|, |\partial\mathfrak{m}| < 2\pi$.

Figure 2.1: An unbounded cone $\Lambda_\mathfrak{m} \subset \mathbb{R}^3$ with a general cross-section $\mathfrak{m} \subset \mathbb{S}^2$ and the circular cone $\Lambda_\mathfrak{b} \subset \mathbb{R}^3$

2.3 2-Manifolds ([KhL22])

In [KhL22] we extended the technique of [AFK17] to a class of compact two-dimensional Riemannian manifolds with boundary. In order to generalize the spectral isoperimetric inequality to this more complicated geometric setting we needed to impose a condition on the curvature.

Let $(\mathcal{M}, g)$ be a (C^∞-)smooth compact two-dimensional, simply-connected Riemannian manifold with C^2-smooth boundary $\partial\mathcal{M}$, equipped with a smooth Riemannian metric g. Such a manifold $\mathcal{M}$ is diffeomorphic to the Euclidean disk and, in particular, its Euler characteristic is equal to 1.

The Robin Laplacian $\mathsf{H}_\beta^\mathcal{M}$ on the manifold $\mathcal{M}$ is defined as in (2.2) with Ω replaced by $\mathcal{M}$ and where Δ should be understood as the Laplace-Beltrami operator. The normal vector is well defined in the respective tangential space for any boundary point of $\mathcal{M}$ and in this perspective the normal derivative is just the scalar product with the metric taken into account of the normal vector and the gradient in the tangential plane evaluated at boundary points.

As in the case of a bounded Euclidean domain, the spectrum of $\mathsf{H}_\beta^\mathcal{M}$ is purely discrete and we denote by $\lambda_1^\beta(\mathcal{M})$ its lowest eigenvalue, the sign of which coincides with the sign of β.

We would like to emphasize that only a few recent results on the optimization of Robin eigenvalues are obtained in the setting of Riemannian manifolds. In [S20], bounds on $\lambda_1^\beta(\mathcal{M})$ in the spirit of the Hersh inequality are proved for compact Riemannian manifolds in any dimension. For positive Robin parameters, $\beta > 0$, a spectral isoperimetric inequality for $\lambda_1^\beta(\mathcal{M})$ in any dimension has been proved very recently in [CGH21] under certain constraints on the curvatures of both the manifold $\mathcal{M}$ and its boundary $\partial\mathcal{M}$. However, the case $\beta < 0$ was not covered by the previous papers.

Let $K_\mathcal{M} \colon \mathcal{M} \to \mathbb{R}$ denote the Gauss curvature of the manifold $\mathcal{M}$. We assume that there exists a constant $K_\circ \geq 0$ such that

$$\sup_{x\in\mathcal{M}} K_\mathcal{M}(x) \leq K_\circ.$$

For $K_\circ = 0$, the manifold $\mathcal{M}$ is of Cartan-Hadamard type and in this case we prove that

$$\lambda_1^\beta(\mathcal{M}) \leq \lambda_1^\beta(\mathcal{B}), \qquad \text{for all } \beta < 0,$$

where $\mathcal{B} \subset \mathbb{R}^2$ is the Euclidean disk having the same perimeter as $\mathcal{M}$. For $K_\circ > 0$, we prove under the assumption $|\mathcal{M}| \leq \frac{2\pi}{K_\circ}$ that

$$\lambda_1^\beta(\mathcal{M}) \leq \lambda_1^\beta(\mathcal{B}^\circ), \qquad \text{for all } \beta < 0,$$

where $\mathcal{B}^\circ \subset \frac{1}{\sqrt{K_\circ}}\mathbb{S}^2$ is the spherical cap in the sphere of radius $R = \frac{1}{\sqrt{K_\circ}}$ having the same perimeter as $\mathcal{M}$ and satisfying $|\mathcal{B}^\circ| \le \frac{2\pi}{K_\circ}$. These additional upper bounds on the areas of $\mathcal{M}$ and $\mathcal{B}^\circ$ can not be removed as a counterexample shows. It remains an open question whether an improvement of these results can be obtained in the case $K_\circ < 0$.

2.4 Magnetic Robin eigenvalues ([KaL22])

In this subsection we address the optimization of the lowest magnetic Robin eigenvalue in two dimensions for the negative boundary parameter. The magnetic field is chosen to be homogeneous.

Let $b \in \mathbb{R}_+$ denote the intensity of the homogeneous magnetic field and let us introduce the vector potential

$$\mathbf{A}(x) = \frac{1}{2}(-x_2, x_1)^\top.$$

The magnetic Robin Laplacian on a bounded sufficiently smooth domain $\Omega \subset \mathbb{R}^2$ is defined as

$$\begin{aligned} \mathsf{H}^\Omega_{\beta,b} u &:= -(\nabla - \mathrm{i}b\mathbf{A})^2 u, \\ \operatorname{dom} \mathsf{H}^\Omega_{\beta,b} &:= \big\{u \in H^2(\Omega) \colon \nu \cdot (\nabla - \mathrm{i}b\mathbf{A})u + \beta u = 0 \,\text{on}\, \partial\Omega\big\}. \end{aligned}$$

The special case $\beta = 0$ corresponds to the magnetic Neumann boundary condition while the case $\beta = \infty$ formally corresponds to the Dirichlet boundary condition. We denote by $\lambda_1^{\beta,b}(\Omega)$ the lowest eigenvalue of $\mathsf{H}^\Omega_{\beta,b}$. It is proved by Erdős in [E96] that the lowest magnetic Dirichlet eigenvalue $\lambda_1^{\infty,b}(\Omega)$ is minimized by the disk among domains of fixed area. The lowest magnetic Neumann eigenvalue $\lambda_1^{0,b}(\Omega)$ is strictly positive and depends on the shape of Ω. Unlike in the non-magnetic case, its optimization is a meaningful problem. It is conjectured by Fournais and Helffer in [FH19] that the lowest magnetic Neumann eigenvalue is maximized by the disk among simply-connected domains of fixed area. This conjecture is substantiated in [FH19] by the analysis of the asymptotic regimes of small and large b.

The optimization of the lowest magnetic Robin eigenvalue has not been considered in the literature before. Intuitively, the case of a positive boundary parameter should be close to the Neumann case for small β and close to the Dirichlet case for large β. In [KaL22] we addressed the case of a negative boundary parameter and obtained a reverse Faber-Krahn-type inequality in this setting. Recall that a domain $\Omega \subset \mathbb{R}^2$ is called centrally symmetric if it is invariant under the mapping $J\colon \mathbb{R}^2 \to \mathbb{R}^2, Jx := -x$. A typical example of a convex centrally symmetric domain is an ellipse. Let the boundary parameter β be negative. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain of the same perimeter as the disk $\mathcal{B} \subset \mathbb{R}^2$ and which is either convex and centrally symmetric or contained in $\mathcal{B}$ upon a suitable translation. Assume that the intensity of the magnetic field satisfies the bound

$$0 < b < \min\{R^{-2}, 4\sqrt{-\beta}R^{-3/2}\}, \tag{2.3}$$

where $R > 0$ is the radius of the disk $\mathcal{B}$. Under all these assumptions the lowest magnetic Robin eigenvalue $\lambda_1^{\beta,b}(\mathcal{B})$ is negative and the inequality

$$\lambda_1^{\beta,b}(\Omega) < \lambda_1^{\beta,b}(\mathcal{B}),$$

holds for $\Omega \ncong \mathcal{B}$.

In fact, we prove a slightly stronger result. The condition (2.3) on b ensures that the lowest magnetic Robin eigenvalue on the disk $\mathcal{B}$ is negative and the respective eigenfunction is a radial function. We can instead assume that $b > 0$ is such that $\lambda_1^{\beta,b}(\mathcal{B}) < 0$ and that the respective eigenfunction is radial. Moreover, the geometric assumptions on the domain Ω can be weakened. In [KaL22] we introduce on Ω and $\mathcal{B}$, having the same perimeter, the distance to the boundary functions $\rho_{\partial\Omega}\colon \Omega \to \mathbb{R}_+$ and $\rho_{\partial\mathcal{B}}\colon \mathcal{B} \to \mathbb{R}_+$. We are able to replace the above geometric assumption on Ω by a subordinacy condition saying that there is a point $x_0 \in \mathbb{R}^2$ such that for $t \in (0, r_{\rm i})$ ($r_{\rm i}$ is the in-radius of Ω) the moment of inertia of the curve $\{x \in \Omega\colon \rho_{\partial\Omega}(x) = t\}$ with respect to x_0 is not greater than that of the curve $\{x \in \mathcal{B}\colon \rho_{\partial\mathcal{B}}(x) = t\}$ with respect to the center of $\mathcal{B}$. This subordinacy condition is satisfied by a convex centrally symmetric Ω and for Ω contained in $\mathcal{B}$ upon a translation. The subordinacy condition is rather generic and it presents an open problem whether there are simply-connected domains that violate this condition.

3 Spectral gap for graphene quantum dots

The Dirac operator defined on a bounded domain of the Euclidean space $\mathbb{R}^2$ attracted a lot of attention in the recent few years. The study of this operator is motivated by the properties of low energy charge carriers in graphene. In the definition of this operator one imposes a general local quantum dot boundary condition, which covers the particular boundary conditions commonly used in the physics literature [AB08]: the so-called *zigzag*, *armchair*, and *infinite mass* boundary conditions. In the following we will focus on the infinite mass boundary condition.

Before going any further, let us rigorously introduce the Dirac operator on a bounded planar domain with infinite mass boundary condition. Let $\Omega \subset \mathbb{R}^2$ be a bounded sufficiently smooth simply-connected domain with the outer unit normal vector $\nu = (\nu_1, \nu_2)^\top$. Recall that the self-adjoint 2×2 Pauli matrices are given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_2 := \begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix}.$$

For the convenience we introduce the collection of matrices $\sigma := (\sigma_1, \sigma_2)$ and define the product $\sigma \cdot \nabla = \sigma_1 \partial_1 + \sigma_2 \partial_2$. Now the Dirac operator on Ω with the infinite mass boundary condition is defined in the Hilbert space $L^2(\Omega; \mathbb{C}^2)$ as

$$\begin{aligned} \mathsf{D}_\Omega u &:= -\mathsf{i}\sigma \cdot \nabla u, \\ \operatorname{dom} \mathsf{D}_\Omega &:= \big\{ u = (u_1, u_2) \in H^1(\Omega; \mathbb{C}^2) \colon u_2 = \mathsf{i}(\nu_1 + \mathsf{i}\nu_2) u_1 \text{ on } \partial\Omega \big\}. \end{aligned} \tag{3.1}$$

According to [BFSV17a] this operator is self-adjoint, its spectrum is discrete and symmetric with respect to the origin, and zero is not an eigenvalue. We denote by $\mu_1(\Omega) > 0$ the smallest positive eigenvalue of D_Ω and refer to it as the principal eigenvalue. In view of the symmetry of the spectrum about the origin the principal eigenvalue determines the size of the spectral gap.

Note that *infinite mass* boundary conditions for the Dirac operator arise when one considers the Dirac operator on the whole Euclidean plane $\mathbb{R}^2$ with an "infinite mass" outside a bounded domain and zero mass inside it. This is mathematically justified in [BCTS19, SV19] (see also [ALMR19] for a three-dimensional version and [MOBP20] for a generalization to any dimension). For this reason, these boundary conditions can be viewed as the relativistic counterpart of Dirichlet boundary conditions for the Laplacian.

It is well known that for partial differential operators defined on domains the shape of the domain manifests in the spectrum. In particular, bounds on the eigenvalues can be given in terms of various geometrical quantities. In many cases, it is also known that the ball (the disk, in two dimensions) optimizes the lowest eigenvalue under reasonable geometric constraints; see the discussion in the beginning of Section 2. In the same spirit, for any convex domain $\Omega \subset \mathbb{R}^2$, it is proven in [PS51, §5.6] and in [FK08, Theorem 2] that a reverse Faber-Krahn-type inequality with a geometric pre-factor for the lowest Dirichlet eigenvalues

$$\lambda_1(\Omega) \leq \frac{|\partial\Omega|}{2r_{\mathrm{i}}|\Omega|} \lambda_1(\mathbb{D}), \tag{3.2}$$

holds where $r_{\mathrm{i}} > 0$ is the inradius of Ω and $\mathbb{D}$ is the unit disk. Related upper bounds for the lowest Dirichlet eigenvalue are obtained e.g. in [PW61, P60].

For the two-dimensional massless Dirac operator D_Ω with infinite mass boundary conditions on a bounded, simply connected, C^2-domain Ω a lower bound on the principal eigenvalue is given in [BFSV17b] and reads in the case of *infinite mass* boundary conditions as

$$\mu_1(\Omega) > \sqrt{\frac{2\pi}{|\Omega|}}. \tag{3.3}$$

This bound is easy to compute and it yields an estimate on the size of the spectral gap. However, it is not intrinsically Euclidean, because the equality in (3.3) is not attained on any $\Omega \subset \mathbb{R}^2$.

One should also mention numerous results in the differential geometry literature, where lower and upper bounds on the principal eigenvalue have been found for Dirac operators on two-dimensional manifolds without boundary (see for instance [B92] and [AF99, B98]). In [R06], manifolds with boundaries are investigated and note that the mentioned CHI (chiral) boundary conditions correspond to our infinite mass boundary conditions. For two-dimensional manifolds, the author of [R06] provides a lower bound on the first eigenvalue which is actually (3.3). We remark that upon passing to the more general setting of manifolds the equality in (3.3) is attained on hemispheres.

Our contributions concern geometric bounds on $\mu_1(\Omega)$. First, we find counterparts of the inequality (3.2) for the principal eigenvalue $\mu_1(\Omega)$ of the Dirac operator D_Ω. To this aim we derive a new variational principle for D_Ω, which can be used to tackle other questions on the spectrum of D_Ω. Second, we support numerically the validity of the Faber-Krahn-type inequality for $\mu_1(\Omega)$.

3.1 Sharp upper bounds on the spectral gap ([ABLO21, LO19])

In this series of papers we obtained counterparts of the inequality (3.2) for the principal eigenvalue of D_Ω. The aim was to get an upper bound on $\mu_1(\Omega)$ in terms of $\mu_1(\mathbb{D})$ and geometric quantities, which turns into inequality provided that Ω is a disk.

In [ABLO21] we proved that

$$\mu_1(\Omega) \leq \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 + 8\pi\mu_1(\mathbb{D})(\mu_1(\mathbb{D})-1)(\pi r_{\rm i}^2 + |\Omega|)}}{2(\pi r_{\rm i}^2 + |\Omega|)}. \tag{3.4}$$

It is not hard to check that for Ω being a disk we get inequality in the above inequality. Combining the inequality (3.4) with the geometric isoperimetric inequality and the definition of the in-radius we obtained a simpler upper bound

$$\mu_1(\Omega) \leq \frac{|\partial\Omega|}{\pi r_{\rm i}^2 + |\Omega|}\mu_1(\mathbb{D}).$$

In order to prove (3.4) we established a new variational principle for the Dirac operator D_Ω, which is inspired by the strategy used to deal in [DES00, DES03] with the Dirac-Coloumb operator. Recall the definition of the Cauchy-Riemann operator $\partial_{\overline{z}} := \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2)$. We consider the following quadratic form

$$\mathfrak{q}_{\mu,0}[u] = 4\|\partial_{\overline{z}} u\|^2_{L^2(\Omega)} - \mu^2\|u\|^2_{L^2(\Omega)} + \mu\|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)}, \quad \operatorname{dom}\mathfrak{q}_{\mu,0} = C^\infty(\overline{\Omega}),$$

and show that this form is semi-bounded and closable. Thus, it defines a self-adjoint operator in the Hilbert space $L^2(\Omega)$. We show that the spectrum of this operator is discrete and denote by $\nu_1^\Omega(\mu)$ its lowest eigenvalue. We prove that $\mu_1(\Omega) = \mu$ if and only if $\nu_1^\Omega(\mu) = 0$. In the case that $\nu_1^\Omega(\mu) = 0$ the respective eigenfunction u solves the oblique problem

$$\begin{cases} -\Delta u = \mu^2 u, & \text{in } \Omega,\\ \partial_\nu u + \mathrm{i}\partial_t u + \mu u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\partial_t u$ is the tangential derivative of u on the boundary $\partial\Omega$. The advantage of this variational method is that we reduce the problem to a one-component operator and circumvent the boundary condition in (3.1).

In [LO19] we obtained a different upper bound on $\mu_1(\Omega)$. Let $f\colon \mathbb{D} \to \Omega$ be a conformal map and let $\kappa_\star > 0$ be the maximum of non-signed curvature of $\partial\Omega$. Then we obtain that

$$\mu_1(\Omega) \leq \left(\frac{2\pi}{\pi r_{\rm i}^2 + |\Omega|}\right)^{1/2} \kappa_\star \|f'\|_{\mathcal{H}^2(\mathbb{D})}\mu_1(\mathbb{D}), \tag{3.5}$$

where $\|\cdot\|_{\mathcal{H}^2(\mathbb{D})}$ stands for the norm in the Hardy space $\mathcal{H}^2(\mathbb{D})$. There are geometric upper bounds on $\|f'\|_{\mathcal{H}^2(\mathbb{D})}$ available in the literature for convex domains [K17] and for nearly circular star-shaped domains [G62]. Combining (3.5) with them we get purely geometric bounds on $\mu_1(\Omega)$.

3.2 Faber-Krahn conjecture ([ABLO21])

We conjecture that the inequality (3.3) can be sharpened in the following way.

Conjecture 3.1. *There holds*

$$\mu_1(\Omega) \geq \sqrt{\frac{\pi}{|\Omega|}}\mu_1(\mathbb{D})$$

where $\mathbb{D}$ is the unit disk. There is equality in the above inequality if and only if Ω is a disk.

This conjecture is equivalent to the fact that among domains of a fixed area the disk is the unique minimizer of the principal eigenvalue of the Dirac operator with the infinite mass boundary condition.

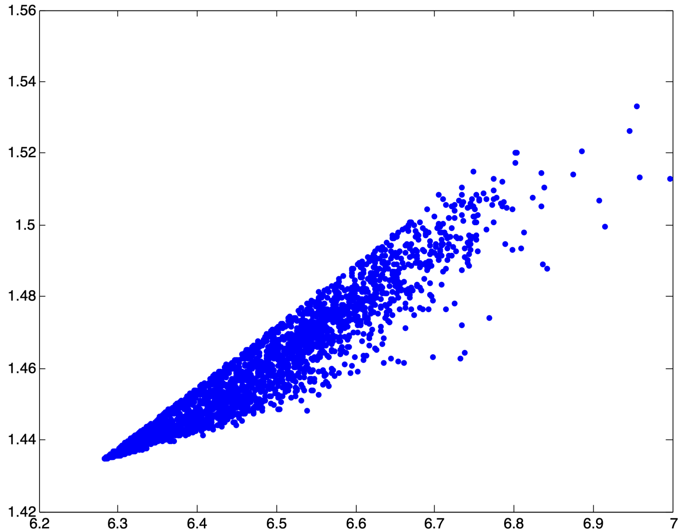

Figure 3.1: Plot of the principal eigenvalue for 2500 domains (with smooth boundary) randomly generated satisfying $|\Omega| = \pi$, as a function of the perimeter.

We have computed the principal eigenvalue for 2500 domains (with smooth boundary) randomly generated satisfying $|\Omega| = \pi$. The corresponding eigenvalues are plotted in Figure 3.1, as a function of the perimeter. We observe that the principal eigenvalue is minimized for the domain which also minimizes the perimeter. By the classical isoperimetric inequality it is well known that for fixed area, the perimeter is minimized by the disk. Thus, these numerical results suggest that the Faber-Krahn type inequality stated in Conjecture 3.1 shall hold for the Dirac operator with infinite mass boundary conditions.

Besides this numerical test we have found that Conjecture 3.1 combined with the one-component variational principle established by us in [ABLO21] yields as a corollary the Bossel-Daners inequality for the Robin Laplacian in two dimensions.

4 δ-Interactions: optimization and spectral asymptotics

Schrödinger operators with δ-interactions supported on hypersurfaces attracted attention in the last three decades; see the review paper [E08], the monograph [EK15], and the references therein. One of the motivations to study these operators is related to the fact that they serve as an idealized model for the Schrödinger operators with regular potentials localized in the vicinity of a hypersurface [BEHL17].

The most efficient way to introduce this operator is via the form method. Let $\Sigma \subset \mathbb{R}^d$, $d \geq 2$, be a Lipschitz hypersurface, which is not necessarily bounded or closed. Let the coupling constant $\alpha > 0$ be fixed. The following symmetric quadratic form in the Hilbert space $L^2(\mathbb{R}^d)$

$$\mathfrak{q}_{\alpha,\Sigma}[u] := \|\nabla u\|^2_{L^2(\Omega;\mathbb{C}^d)} - \alpha \|u|_\Sigma\|^2_{L^2(\Sigma)}, \qquad \operatorname{dom} \mathfrak{q}_{\alpha,\Sigma} := H^1(\mathbb{R}^d), \tag{4.1}$$

is closed, densely defined, and semi-bounded. Hence, it defines a self-adjoint operator $\mathsf{H}_{\alpha,\Sigma}$ in the Hilbert space $L^2(\mathbb{R}^d)$. In the case that $\Sigma \subset \mathbb{R}^d$ is a bounded closed sufficiently smooth hypersurface, which splits the Euclidean space into a bounded domain $\Omega_+ \subset \mathbb{R}^d$ and an exterior domain $\Omega_- \subset \mathbb{R}^d$, the operator $\mathsf{H}_{\alpha,\Sigma}$ can be explicitly characterised as

$$\begin{aligned} \mathsf{H}_{\alpha,\Sigma} u &= (-\Delta u_+) \oplus (-\Delta u_-), \\ \operatorname{dom} \mathsf{H}_{\alpha,\Sigma} &= \Big\{ u \in H^2(\mathbb{R}^d \setminus \Sigma) \colon u_+|_\Sigma = u_-|_\Sigma,\ \partial_{\nu_+} u_+|_\Sigma + \partial_{\nu_-} u_-|_\Sigma = \alpha u|_\Sigma \Big\}, \end{aligned}$$

where $u_\pm := u|_{\Omega_\pm}$ and $\partial_{\nu_\pm} u_\pm|_\Sigma$ stands for the normal derivative of $u_\pm$ with the normal pointing outwards of $\Omega_\pm$. The spectrum of $\mathsf{H}_{\alpha,\Sigma}$ depends in a non-trivial way on α and Σ and exploring this connection is a topic of permanent interest.

For compact Σ the essential spectrum of $\mathsf{H}_{\alpha,\Sigma}$ coincides with the interval $[0,\infty)$ and there are finitely many negative eigenvalues. In two dimensions there is at least one negative eigenvalue for all $\alpha > 0$. In higher space dimensions there is a critical value $\alpha_\star = \alpha_\star(\Sigma) > 0$ such that $\sigma_{\rm d}(\mathsf{H}_{\alpha,\Sigma}) \neq \varnothing$ if, and only if $\alpha > \alpha_\star$. The geometry of Σ manifests in a non-trivial in the asymptotics of eigenvalues of $\mathsf{H}_{\alpha,\Sigma}$ in the limit $\alpha \to +\infty$. It turns out that the sub-leading term in these asymptotic expansions can be expressed through eigenvalues of a Schrödinger operator on Σ with a potential given in terms of the curvatures of Σ (see *e.g.* [E03, EP14, EY02]). For fixed $\alpha > 0$, it is of interest to optimize the lowest eigenvalue of $\mathsf{H}_{\alpha,\Sigma}$ among surfaces Σ that fulfil certain geometric constraints. In particular, it is proved in [EHL06] that in two dimensions the lowest negative eigenvalue of $\mathsf{H}_{\alpha,\Sigma}$ is maximized by a circle among all sufficiently smooth contours without self-intersections of fixed length.

For non-compact Σ the situation is qualitatively different. In particular, for Σ being a local deformation of a hyperplane or more generally asymptotically flat, the essential spectrum of $\mathsf{H}_{\alpha,\Sigma}$ coincides with the interval $[-\frac{\alpha^2}{4}, +\infty)$. Existence of bound states below the point $-\frac{\alpha^2}{4}$ is a delicate question. It is shown in [EI01] that δ-interaction supported on an asymptotically straight unbounded curve in $\mathbb{R}^2$ induces at least one bound state below the threshold of the essential spectrum $-\frac{\alpha^2}{4}$ provided that the interaction support does not coincide with a straight line. In three dimensions a counterpart of this result for asymptotically flat surfaces is proved in [EK03] under the assumption that α is sufficiently large. The case of moderate α in three dimensions remains open.

We address several related questions on the asymptotics of eigenvalues and spectral optimization for $\mathsf{H}_{\alpha,\Sigma}$ and its generalizations mainly in two and three dimensions. First, we investigate the three-dimensional Schrödinger operator with a δ-interaction supported on an unbounded conical surface. In this geometric setting we characterise the essential spectrum, compute the asymptotics of the discrete spectrum, and obtain an optimization result for the lowest eigenvalue. Second, we analyse the existence and asymptotics of a bound state of $\mathsf{H}_{\alpha,\Sigma}$ for Σ being a weak local deformation of a plane in three dimensions. Furthermore, we discuss optimization of the lowest eigenvalue of $\mathsf{H}_{\alpha,\Sigma}$ in two dimensions for Σ being an arc with two endpoints and consider optimization of the lowest eigenvalue in some related settings. Finally, we modify the operator $\mathsf{H}_{\alpha,\Sigma}$ by adding a homogeneous magnetic field in the kinetic energy term. In this last setting our results concern the asymptotics of accumulation of the discrete spectra at the Landau levels.

4.1 Conical surfaces ([BEL14, EL17, LO16])

Let $\Sigma_\theta \subset \mathbb{R}^3$ be the conical surface defined as

$$\Sigma_\theta := \Big\{ \big(x_1, x_2, \cot\theta (x_1^2 + x_2^2)^{1/2}\big) \colon (x_1, x_2) \in \mathbb{R}^2 \Big\},$$

where $\theta \in (0, \pi/2)$ is the aperture of Σ_θ. We consider the Schrödinger operator $\mathsf{H}_{\alpha,\Sigma_\theta}$ with the δ-interaction of strength $\alpha > 0$ supported on Σ_θ. In [BEL14] we established that the essential spectrum of this operator coincides with the interval $[-\frac{\alpha^2}{4}, +\infty)$ and that its discrete spectrum is infinite. Moreover, we proved that the same properties persist for δ-interaction of strength $\alpha > 0$

supported on a local deformation of the conical surface Σ_θ. We also obtained an explicit upper bound on the eigenvalues of $\mathsf{H}_{\alpha,\Sigma_\theta}$.

A more detailed analysis of the discrete spectrum of $\mathsf{H}_{\alpha,\Sigma_\theta}$ is carried out in [LO16]. We establish that the eigenfunctions corresponding to the discrete eigenvalues of $\mathsf{H}_{\alpha,\Sigma_\theta}$ are all rotationally invariant with respect to the x_3-axis and that the eigenvalues of $\mathsf{H}_{\alpha,\Sigma_\theta}$ are non-decreasing in the aperture θ. Moreover, we find the spectral asymptotics for $\mathsf{H}_{\alpha,\Sigma_\theta}$. Recall that the counting function $\mathbb{R}_+ \ni E \mapsto \mathcal{N}_{-\frac{\alpha^2}{4}-E}(\mathsf{H}_{\alpha,\Sigma_\theta})$ of the discrete spectrum of $\mathsf{H}_{\alpha,\Sigma_\theta}$ is defined for any fixed $E > 0$ by the number of eigenvalues of the operator $\mathsf{H}_{\alpha,\Sigma_\theta}$ lying in the interval $(-\infty, -\frac{\alpha^2}{4} - E)$ with multiplicities taken into account. In [LO16] we proved that

$$\mathcal{N}_{-\frac{\alpha^2}{4}-E}(\mathsf{H}_{\alpha,\Sigma_\theta}) \sim \frac{\cot\theta}{4\pi}|\ln E|, \qquad E \to 0^+.$$

Besides the three-dimensional setting we address in [LO16] the counterpart of this problem in space dimensions $d \geq 4$. In the latter setting we obtain that the essential spectrum is again $[-\frac{\alpha^2}{4}, \infty)$ while the discrete spectrum is empty.

Finally, in [EL17] we consider δ-interactions supported on more general non-circular conical surfaces in three dimensions. Let $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth loop on the unit sphere $\mathbb{S}^2$. Then we define the conical surface with the cross-section $\mathcal{T}$ by

$$\Sigma(\mathcal{T}) := \{r\mathcal{T} \colon r > 0\}.$$

It follows from the results of [BP16] that $-\frac{\alpha^2}{4}$ is the lowest point of the essential spectrum of the Schrödinger operator $\mathsf{H}_{\alpha,\Sigma(\mathcal{T})}$ with δ-interaction of strength $\alpha > 0$ supported on $\Sigma(\mathcal{T})$. Let us denote by $\lambda_1^\alpha(\Sigma(\mathcal{T}))$ the lowest spectral point of $\mathsf{H}_{\alpha,\Sigma(\mathcal{T})}$. In [EL17] we prove the following isoperimetric inequality under the assumption $|\mathcal{T}| < 2\pi$

$$\lambda_1^\alpha(\Sigma(\mathcal{T})) \leq \lambda_1^\alpha(\Sigma(\mathcal{C})), \qquad \text{for all } \alpha > 0, \tag{4.2}$$

where $\mathcal{C} \subset \mathbb{S}^2$ is the circle of the same length as $\mathcal{T}$. This result, in particular, implies that at least for short cross-sections ($|\mathcal{T}| < 2\pi$) the lowest spectral point of $\mathsf{H}_{\alpha,\Sigma(\mathcal{T})}$ is indeed a discrete eigenvalue below the bottom of the essential spectrum $-\frac{\alpha^2}{4}$. In paper [EL17] we also analyse the situation of a bounded conical surface, which can be defined as the intersection of an unbounded conical surface $\Sigma(\mathcal{T})$ with a ball of certain radius centred at the origin. In the latter setting we again obtain an isoperimetric inequality for the lowest eigenvalue similar to (4.2).

4.2 Weak local deformations ([EKL18])

As it was already mentioned it remains an open problem to prove that in three dimensions δ-interaction of strength $\alpha > 0$ supported on a local deformation of a plane induces at least one bound state below the bottom of the essential spectrum $-\frac{\alpha^2}{4}$. Motivated by this open problem we considered in [EKL18] δ-interactions supported on weak local deformations of the plane.

Let us first describe the geometric setting. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^2-smooth compactly supported function ($f \not\equiv 0$). Consider the hypersurface

$$\Sigma_\beta := \{(x, \beta f(x))\colon x \in \mathbb{R}^2\}, \qquad \beta > 0.$$

The essential spectrum of $\mathsf{H}_{\alpha,\Sigma_\beta}$ coincides with the interval $[-\frac{\alpha^2}{4}, \infty)$. It follows from [EK03] that for any $\beta > 0$ and for all sufficiently large $\alpha > 0$ there is at least one bound state for $\mathsf{H}_{\alpha,\Sigma_\beta}$ below $-\frac{\alpha^2}{4}$. Our aim was to consider the case of fixed $\alpha > 0$ and sufficiently small $\beta > 0$.

We prove that for any fixed $\alpha > 0$ and for all sufficiently small $\beta > 0$ the discrete spectrum of $\mathsf{H}_{\alpha,\Sigma_\beta}$ is non-empty and consists of exactly one simple eigenvalue, which we denote by $\lambda_1^\alpha(\Sigma_\beta)$. Our next result concerns the asymptotics of this eigenvalue. Let us introduce the quantity

$$\mathcal{D}_{\alpha,f} := \int_{\mathbb{R}^2} |p|^2 \left(\alpha^2 - \frac{2\alpha^3}{\sqrt{4|p|^2 + \alpha^2} + \alpha}\right) |\widehat{f}(p)|^2 \mathsf{d}p > 0,$$

where $\widehat{f}$ is the Fourier transform of f. We obtain that

$$\lambda_1^\alpha(\Sigma_\beta) = -\frac{\alpha^2}{4} - \exp\left(-\frac{16\pi}{\mathcal{D}_{\alpha,f}\beta^2}\right)\big(1 + o(1)\big), \qquad \beta \to 0^+.$$

4.3 Optimization for arcs and loops ([EL21, L19, L21])

Motivated by the result [EHL06] on the optimization of the lowest eigenvalue for δ-interaction supported on a loop in $\mathbb{R}^2$, we obtained spectral isoperimetric inequalities in several related settings using different techniques.

In [L19] we optimize the lowest eigenvalue of the two-dimensional Schrödinger operator $\mathsf{H}_{\alpha,\Sigma}$ with δ-interaction supported on an open C^2-smooth arc $\Sigma \subset \mathbb{R}^2$ with two endpoints. Let $\lambda_1^\alpha(\Sigma)$ be the lowest negative eigenvalue of this operator. We prove that

$$\lambda_1^\alpha(\Sigma) \le \lambda_1^\alpha(\Gamma), \qquad \text{for all}\, \alpha > 0,$$

where $\Gamma \subset \mathbb{R}^2$ is a line segment of the same length as Σ. The equality is possible only if Γ and Σ are congruent. The constraint of the fixed length can be replaced by fixing the endpoints of the arc and the maximizer is the line segment connecting them.

We provide in [EL21] an alternative proof of the main result of [EHL06] without using the Birman-Schwinger principle. Furthermore, we apply in [EL21] this technique to a more general class of operators. Let $\Sigma \subset \mathbb{R}^2$ be a closed C^2-smooth loop parametrized by the unit speed map $\sigma\colon [0, L] \to \mathbb{R}^2$ with the normal vector ν pointing outwards a bounded domain surrounded by Σ. Let the parameters $d_-, d_+ > 0$ be such that the mapping

$$[0, L) \times [-d_-, d_+] \ni (s, t) \mapsto \sigma(s) + t\nu(\sigma(s))$$

is injective. This map defines parallel coordinates (s, t) on its range being the curved strip in $\mathbb{R}^2$. We restrict our attention to measures μ of a special structure given in parallel coordinates by

$$\mathsf{d}\mu = (1 + \kappa(s)t)\mathsf{d}s\mathsf{d}\mu_\perp(t), \tag{4.3}$$

where κ is the curvature of Σ and $\mu_\perp$ is a finite measure on the interval $[-d_-, d_+]$. The self-adjoint operator H_μ in the Hilbert space $L^2(\mathbb{R}^2)$ associated to the formal differential expression $-\Delta - \mu$ can be rigorously introduced via the quadratic form

$$H^1(\mathbb{R}^2) \ni u \mapsto \|\nabla u\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} - \int_{\mathbb{R}^2} |u|^2 \mathsf{d}\mu.$$

The essential spectrum of H_μ is $[0, \infty)$ and it has at least one negative eigenvalue. The case of δ-interaction of strength $\alpha > 0$ supported on Σ corresponds to the choice $\mu_\perp = \alpha\delta_0$, where δ_0 is the Dirac δ-function supported at the origin. By choosing the transversal measure $\mathsf{d}\mu_\perp = V_\perp(x)\mathsf{d}x$ with real-valued non-negative $V_\perp \in L^\infty((-d_-, d_+))$ we appear in the setting resembling the soft quantum waveguide proposed in [E20].

We fix the transversal measure $\mu_\perp$ and optimize the lowest eigenvalue $\lambda_1(\mu)$ of H_μ with respect to the shape of Σ. We prove that

$$\lambda_1(\mu) \le \lambda_1(\mu_\circ)$$

where $\mu_\circ$ is the measure on $\mathbb{R}^2$ of the type (4.3) having the same transversal part $\mu_\perp$ as μ, but constructed on the circle of the same length as Σ.

Along with the above optimization problem we optimize $\lambda_1(\mu)$ for fixed Σ under variation of the transversal measure $\mu_\perp$. We prove that

$$\lambda_1(\mu) \ge \lambda_1(\mu_\star)$$

where $\mu_\star$ is the measure on $\mathbb{R}^2$ of the type (4.3) constructed on the same loop Σ with the transversal measure being $\alpha\delta_{t_\star}$ for certain $t_\star \in [-d_-, d_+]$ with $\alpha = \mu_\perp([-d_-, d_+])$. In other words under the constraint of fixed total transversal measure of the interval $[-d_-, d_+]$ the minimizer of the lowest eigenvalue turns out to be a δ-interaction supported on a certain level curve of the distance function to the loop Σ. We provide examples showing that the optimal position $t_\star$ need not coincide with an endpoint of the interval $[-d_-, d_+]$.

Finally, in [L21] we optimize the lowest eigenvalue of the Schrödinger operator with a δ'-interaction supported on a C^2-smooth loop Σ in $\mathbb{R}^2$. The δ'-interaction is studied along with δ-interactions and the Schrödinger operator with δ'-interaction supported on hypersurface were first rigorously introduced in [BLL13]. Suppose that Σ splits the Euclidean plane into a bounded domain $\Omega_+ \subset \mathbb{R}^2$ and an unbounded exterior domain $\Omega_- \subset \mathbb{R}^2$. For a function $u \in L^2(\mathbb{R}^2)$ we recall the notation $u_\pm := u|_{\Omega_\pm}$. For $\omega > 0$, consider the quadratic form in the Hilbert space $L^2(\mathbb{R}^2)$

$$H^1(\mathbb{R}^2 \setminus \Sigma) \ni u \mapsto \|\nabla u_+\|^2_{L^2(\Omega_+;\mathbb{C}^2)} + \|\nabla u_-\|^2_{L^2(\Omega_-;\mathbb{C}^2)} - \omega\|u_+|_\Sigma - u_-|_\Sigma\|^2_{L^2(\Sigma)}.$$

This quadratic form defines a self-adjoint operator $\mathsf{H}'_{\omega,\Sigma}$ in the Hilbert space $L^2(\mathbb{R}^2)$. The operator $\mathsf{H}'_{\omega,\Sigma}$ is regarded as Schrödinger operator with δ'-interaction of strength $\beta = 1/\omega$ supported on Σ. As in the case of δ-interactions the essential spectrum of $\mathsf{H}'_{\omega,\Sigma}$ coincides with the interval $[0, \infty)$ and the negative discrete spectrum is non-empty. We denote by $\mu_1^\omega(\Sigma)$ the lowest negative eigenvalue of $\mathsf{H}'_{\omega,\Sigma}$. We prove that

$$\mu_1^\omega(\Sigma) \le \mu_1^\omega(\mathcal{C}), \qquad \text{for all } \omega > 0,$$

where $\mathcal{C}$ is a circle of the same length as Σ.

4.4 Landau Hamiltonians with δ-interactions ([BEHL20])

In [BEHL20] we study the Landau Hamiltonian perturbed by a δ-interaction supported on a curve. This operator is defined by a quadratic form similar to the one in (4.1) with the magnetic gradient in the kinetic energy term. Recall that the vector potential of the homogeneous magnetic field of intensity $B \ge 0$ is defined by

$$\mathbf{A} = \frac{1}{2}B(-x_2, x_1)^\top.$$

We introduce the magnetic gradient by

$$\nabla_\mathbf{A} := \mathrm{i}\nabla + \mathbf{A}.$$

Let $\Sigma \subset \mathbb{R}^2$ be the boundary of a compact $C^{1,1}$-domain and let $\alpha \in L^\infty(\Sigma)$ be real-valued. We consider the quadratic form

$$\begin{aligned} \mathfrak{q}^B_{\alpha,\Sigma}[u] &:= \|\nabla_{\mathbf{A}} u\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + (\alpha u|_\Sigma, u|_\Sigma)_{L^2(\Sigma)}, \\ \operatorname{dom} \mathfrak{q}^B_{\alpha,\Sigma} &:= \{u\colon u, |\nabla_{\mathbf{A}} u| \in L^2(\mathbb{R}^2)\}. \end{aligned}$$

This quadratic form defines a self-adjoint operator $\mathsf{H}^B_{\alpha,\Sigma}$ in the Hilbert space $L^2(\mathbb{R}^2)$, which we regard as the Landau Hamiltonian with δ-interaction of strength α supported on Σ. Here, we have chosen a slightly different convention. Namely, the interaction term in the quadratic form comes with a different sign and the interaction strength is varying along the support of interaction.

In the case $B = 0$ the Landau Hamiltonian $\mathsf{H}^0_{\alpha,\Sigma}$ is essentially the Schrödinger operator with δ-interaction supported on Σ defined via the form in (4.1). While in the case that $\alpha \equiv 0$ the operator $\mathsf{H}^B_{0,\Sigma}$ is just the usual Landau Hamiltonian, which we denote by H^B.

We show that the essential spectrum of $\mathsf{H}^B_{\alpha,\Sigma}$ consists of Landau levels.

$$\sigma_{\rm ess}(\mathsf{H}^B_{\alpha,\Sigma}) = \{B(2q+1)\colon q \in \mathbb{N}_0\}.$$

It is well known that perturbations of the Landau Hamiltonian H^B can generate accumulation of discrete eigenvalues to the Landau levels. For additive perturbations of H^B by an electric potential this was shown by Raikov in [R90]. More recently similar results were proved by Pushnitski and Rozenblum in [PR07] for Landau Hamiltonians on exterior domains with Dirichlet boundary conditions. We analyse this phenomenon for the operator $\mathsf{H}^B_{\alpha,\Sigma}$.

We prove spectral asymptotics if $\operatorname{supp}\alpha$ is a C^∞-smooth arc $\Gamma \subset \Sigma$ and α is uniformly positive (uniformly negative) in the interior of Γ. In this asymptotics enters the logarithmic capacity of Γ. Recall that the logarithmic energy of a measure μ on $\mathbb{R}^2$ is given by

$$I(\mu) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}\mathsf{d}\mu(x)\mathsf{d}\mu(y).$$

The logarithmic capacity of a compact set $\mathcal{K} \subset \mathbb{R}^2$ is defined by

$$\mathrm{Cap}\,(\mathcal{K}) := \sup\left\{e^{-I(\mu)}\colon \mu \text{ measure on } \mathbb{R}^2,\ \operatorname{supp}\mu \subset \mathcal{K}, \mu(\mathcal{K}) = 1\right\}.$$

If, e.g., $\alpha > 0$ inside the C^∞-smooth arc $\Gamma = \operatorname{supp}\alpha$ then the discrete eigenvalues (counted with multiplicities) of $\mathsf{H}^B_{\alpha,\Sigma}$ in the interval $(B(2q+1), B(2q+2)]$,

$q \in \mathbb{N}_0$, form a sequence $\lambda_1^+(q) \geq \lambda_2^+(q) \geq \cdots \geq B(2q+1)$ with the asymptotic behavior

$$\lim_{k \to \infty} \Big(k!\Big(\lambda_k^+(q) - B(2q+1)\Big)\Big)^{1/k} = \frac{B}{2}\big(\mathrm{Cap}\,(\Gamma)\big)^2.$$

For sign-changing α and for less smooth $\mathrm{supp}\,\alpha$ we obtain estimates on the accumulation rate of the eigenvalues towards the Landau levels.

My articles on which the thesis is based

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