Characteristics of main research directions investigated at the institute and the achievements 2010−2014

Institute	Institute of Mathematics of the CAS, v. v. i.

The principal mission of the Institute of Mathematics (IM) is to support fundamental research in the fields of mathematics and its applications, and to provide necessary infrastructure for research. The IM contributes to raising the level of knowledge and education and to utilising the results of scientific research in practice. It acquires, processes and disseminates scientific information, issues scientific and professional publications (monographs, journals, proceedings, preprints, etc.). In cooperation with Czech universities, the IM carries out doctoral study programmes and provides training for young scientists. The IM promotes international cooperation, including the organisation of joint research projects with foreign partners and participation in exchange programmes and international research networks. It has been one of the most successful Czech research institutions in the European Research Council (ERC) Grant competition. The IM organises scientific meetings, conferences and seminars on the national and international levels. Its experts regularly give specialised courses at foreign universities, and are invited as plenary or key note speakers in international congresses and conferences.

Research in the Institute focuses on mathematical analysis (differential equations, numerical analysis, functional analysis, theory of function spaces), mathematical physics, mathematical logic, complexity theory, combinatorics, set theory, numerical algebra, topology (general and algebraic), optimization and control, and differential geometry. The structure of IM consists of the following six research teams, and a small group for Didactics of Mathematics, which is not registered as a research team for this evaluation.

Research Report of the team in the period 2010–2014

Institute	Institute of Mathematics of the CAS, v. v. i.

Scientific team	Algebra, Geometry and Mathematical Physics

Since the beginning of 20th century, mathematical physics combines methods of seemingly separate areas of mathematics – differential geometry (relativity), functional analysis (quantum mechanics), complex functions (conformal field theory, vertex operator algebras), algebraic topology and geometry (string field theory), symplectic geometry (mechanics), topological field theory (categories), group theory and representation (quantum groups), etc.

All the above mentioned fields have merged into mainstream mathematics of today. This fact is being reflected in the structure of several leading mathematical centres around the world, for instance in the Department of Applied Mathematics and Theoretical Physics (DAMTP), one of the two departments of the Faculty of Mathematics of Cambridge University.

We felt that a department of this type is missing in the Czech Republic, all the more so since there were several mathematicians working in these areas, scattered thorough the existing departments of the Institute and informally communicating with each other. Moreover, some promising students with mathematical or physical background have emerged. For this reason, a new department has been established in the Institute to bring these scientists and students together, combining their efforts and amplifying the efficiency of their work by synergy. Thus, the department exists formally since December 2014 only but the team can naturally build on the previous work of its members.

1. Mathematical relativity

These topics which fall, in general, on the interface between (Lorentzian) differential geometry and mathematical physics, are studied by V. Pravda, A. Pravdová, M. Ortaggio and T. Málek.

GHP formalism

The Newman-Penrose and Geroch-Held-Penrose (GHP) formalisms are standard mathematical tools of mathematical relativity in four dimensions. In collaboration with H. Reall and M. Durkee from the DAMTP, Cambridge, we generalized the GHP formalism to the case of arbitrary dimension [348189]. This allows us to study various properties of higher-dimensional spacetimes, in particular spacetimes with a preferred null direction.

In fact, our new formalism has been already used in such distinct contexts as stability of various black hole solutions, classifications of near horizon geometries, numerical relativity, uniqueness of certain exact solutions of the Einstein equations, asymptotic properties of spacetimes etc. So far, the paper [348189] has gathered 48 citations on arXiv and 36 in the Web of Science database.

This paper is a natural continuation of our previous works [106853] and [80186], where we have developed the Newman-Penrose formalism in arbitrary dimension. Since both groups (Prague and Cambridge) planned to use such formalism in the future research we joined our forces to develop this formalism.

Einstein spacetimes

We studied various aspects of Einstein spacetimes in higher dimensions.

- **Kerr-Schild spacetimes**
  The well-known Kerr-Schild ansatz is compatible with important exact solutions of the Einstein equations in four and higher dimensions (this includes, e.g., the Kerr and Myers-Perry black holes and certain Kundt gravitational waves). In [359548], we studied Einstein spacetimes that can be cast into the Kerr-Schild form. We proved that all such spacetimes are algebraically special and we studied geometric properties of the Kerr-Schild vector. We showed that it obeys the optical constraint, which subsequently led to the studies of the Goldberg-Sachs theorem mentioned below. These results also allowed us to partially integrate some of the Ricci identities. In [434069], a more general „extended Kerr-Schild“ ansatz was studied.

- **Generalization of the Goldberg-Sachs theorem to higher dimensions**
  In four dimensions, the Goldberg-Sachs theorem is a well-known theorem representing a first step towards the full integration of the vacuum Einstein equations. Building on our previous results, we studied a generalization of this theorem to five dimensions in collaboration with H. Reall (DAMTP, Cambridge) [380306]. We have derived necessary conditions for the geometrical properties of a preferred null geodetic congruence (mWAND) in algebraically special five-dimensional Einstein spacetimes. In [391346], we studied a generalization of this theorem to dimensions higher than five for spacetimes admitting a non-twisting multiple Weyl aligned null direction.
  In [380306], we combined the expertise of H. Reall on the “geodetic” part of the Goldberg-Sachs theorem [DR09] with our earlier partial results on the “shearfree” part [106853], [80186]. We have proven the theorem using the higher-dimensional Newman-Penrose and GHP formalisms (see above).
  [DR09] M. Durkee, H. S. Reall, *A Higher-dimensional generalization of the geodesic part of the Goldberg-Sachs theorem*, CQG **26** (2009) 245005
  (CQG = Classical and Quantum Gravity)

- **Asymptotic properties of Einstein spacetimes**
  In [434752], we have determined the leading order fall-off behaviour of the Weyl tensor in higher-dimensional Einstein spacetimes (with and without a cosmological constant) as infinity is approached along a congruence of null geodesics. The null congruence is assumed to “expand” in all directions near infinity (but it is otherwise generic), which includes, in particular, asymptotically flat spacetimes. In contrast to the well-known four-dimensional peeling property, the fall-off rate of various Weyl components depends substantially on the chosen boundary conditions and is also influenced by the presence of a cosmological constant. The leading component is always algebraically special but, in various cases, it can be of type N, III, or II.

- **Einstein spacetimes with a warped extra dimension**
  We studied a class of higher-dimensional warped Einstein spacetimes [359288], which is of interest in the description of brane-world models and warped black strings. In particular, we clarified the relation between various coordinate systems used in the literature and analyzed the Weyl tensor type in the various possible cases. We also studied the presence of spacetime singularities. The results of this paper can be used to generate new solutions with certain properties via a suitable ansatz.

- **Type III and N Einstein spacetimes**
  Einstein spacetimes of type III and N, often interpreted as “radiative” in four dimensions, are qualitatively different in higher dimensions. For this reason, in [348265], we studied their asymptotic properties along a congruence of (geometrically preferred) null directions and contrasted it with the standard peeling behaviour of the Weyl tensor. We also discussed the presence of curvature singularities. As a side-result, we also solved the Sachs equations governing the evolution of the optical matrix of geodetic WANDs in arbitrary dimension in several cases of interest in applications.

Generalized theories of gravity

- **Quadratic gravity**
  In perturbative quantum gravity, corrections have to be added to the Einstein-Hilbert action. One class of such generalized theories of gravity is the so called quadratic gravity (QG). The resulting field equations are considerably more complicated than the Einstein equations, nevertheless one of the first exact solutions of this theory (obtained using the Kerr-Schild ansatz mentioned above) was found in [GGST11]. In [362717], we proved that in an arbitrary dimension all Einstein spacetimes of the Weyl type N are vacuum solutions of QG. We also found a class of Einstein type III and of non-Einstein type N vacuum solutions of QG.
  [GGST11] I. Gullu, M. Gurses, T. C. Sisman, B. Tekin, *AdS Waves as Exact Solutions to Quadratic Gravity*, Phys. Rev. D **83** (2011) 084015

- **Universal spacetimes**
  Since the final form of the quantum gravity corrections to the Einstein equations is at present unknown, it is of interest to study the so called universal spacetimes. These spacetimes are vacuum solutions to all possible diffeomorphism-invariant generalized theories of gravity, since they solve the vacuum field equations derived from *any* Lagrangian formed from the metric, the Riemann tensor and its derivatives.
  First examples of such spacetimes were identified as classical solutions to the string theory in [AK89] and [HS90], and as spacetimes with vanishing quantum corrections in [CGHP08]. In [433091], we started a systematic study of universal spacetimes and we have proven that all such spacetimes have constant curvature invariants. For type N spacetimes, we have found simple necessary and sufficient conditions for universality and provided various explicit examples. For type III, we have found sufficient conditions for universality and also provided examples.
  In various proofs in this paper, we used results and methods that we originally developed for spacetimes with vanishing curvature invariants [175182] and [106944]. We also extensively used our higher-dimensional GHP formalism. In

2008, S. Hervik co-authored a seminal paper [CGHP08] on spacetimes with vanishing quantum corrections (which are in fact universal). The combination of these two sets of results and methods was the key starting point for this work.
[AK89] D. Amati and C. Klimčík. *Nonperturbative computation of the Weyl anomaly for a class of nontrivial backgrounds*. Phys. Lett. B **219** (1989) 443–447
[HS90] G. T. Horowitz and A. R. Steif, Phys. Rev. Lett., **64** (1990) 260–263
[CGHP08] A. Coley, G. W. Gibbons, S. Hervik, and C. N. Pope, *Metrics with vanishing quantum corrections*, CQG **25** (2008) 145017

Classification of spacetimes

- **Refinements of the Weyl tensor classification in five dimensions**
The Petrov-Penrose classification of the Weyl tensor in four dimensions is one important tool of mathematical relativity. In our previous research, we, in collaboration with A. Coley and R. Milson (Dalhousie University, Halifax, Canada), developed in [106805] algebraic classification of arbitrary tensors in arbitrary dimension which is for the Weyl tensor in four dimensions equivalent to the Petrov classification. In [380712], M. Ortaggio with collaborators refined the classification in the special case of five-dimensional spacetimes, in particular for the algebraically special types II, D, III and N. Building on previous work by [CH09] a first refinement is provided by the notion of "spin type". A second proposed refinement consists in the Segre types of the Weyl operator acting on the bivector space. We also examined the intersections between these various classification schemes. The contribution of M. Ortaggio focused, in particular, on the results and discussion related to the types II and D.

  [CH09] A. Coley and S. Hervik, *Higher dimensional bivectors and classification of the Weyl operator*, CQG **27** (2009) 015002

- **Minimal tensors and purely electric or magnetic spacetimes**
We defined the electric and magnetic parts of a tensor in terms of transformation properties under time reversal [395526]. We applied this to the Weyl tensor, in relation with the null alignment classification [106805]. We extended some known four-dimensional results and pointed out new features of higher dimensions. We made a connection with the alignment theorem [H11], with the theory of "minimal" tensors [RS90] and with previous work by [S00]. Paper [395526] was included in the selection "IOP Select" by the Editorial Board of *Classical and Quantum Gravity*.

  We combined the expertise of S. Hervik in group theory and minimal tensors and that of L. Wylleman in four-dimensional "electric and magnetic" spacetimes with the null alignment methods developed and studied by our Prague group and collaborators [106805], [21972]. In particular, techniques previously used in the study of static and warped spacetimes [85303] as well as Bel-Debever like criteria [330029] were employed.
  [H11] S. Hervik, *A spacetime not characterized by its invariants is of aligned type II*, CQG **28** (2011) 215009
  [RS90] R. W. Richardson and P. Slodowy, *Minimum vectors for real reductive algebraic groups*, J. London Math. Soc. **42** (1990) 409–429
  [S00] J. M. M. Senovilla, *Super-energy tensors*, CQG **17** (2000) 2799–2841

Other topics

- **Asymptotic behaviour of Maxwell fields**
  Paper [437500] is the electromagnetic counterpart of the work [434752] mentioned above. The “peeling” behaviour of Maxwell fields has been determined in various backgrounds that include, in particular, (asymptotically) flat and (anti-)de Sitter spacetimes. General *p*-form fields, of possible interest in supergravity and string theory, have also been discussed. This also stimulated subsequent investigation of certain spacetimes where gravity is coupled to *p*-forms (see e.g. very recent work [441470]).

- **Topical review**
  The work [387097] is an invited (yet refereed) review article for the journal Classical and Quantum Gravity. Here, we summarize the general formalism and main results related to algebraically special spacetimes in higher dimensions, based mainly on our results obtained with various collaborators since 2004.

2. Algebraic topology and geometry

This is research interest of M. Markl who has focused in 2010–2014 mainly on the following topics.

- **Deligne's conjecture**
  Together with M. Batanin (Macquarie University, Australia) and C. Berger (Université de Nice, France) M. Markl proved various forms of the Deligne conjecture about the structure of natural operations on the Hochschild cochain complex. The results were published in several papers [377482], [381389], [433876]; the most important one, [377482] appeared in Advances in Mathematics. M. Markl contributed by detailed combinatorial analysis of tree complexes, by identification of various versions of the operad of natural operations, and by establishing homotopy equivalences among them.

- **Combinatorial differential geometry**
  M. Markl, together with J. Janyška (Masaryk University, Brno), pursued his idea of “combinatorial geometry” based on graphs complexes. Using this approach, they improved some classical structure theorems of Riemannian geometry, both in the torsion and torsion-free case [362692], [382840]. M. Markl's contribution consisted in his deep knowledge of the graph complexes techniques and his idea of using it in the context of differential geometry.

- **Koszul hierarchy**
  M. Markl gave a surprisingly simple explanation of the origin of Koszul braces, both for the commutative and the non-commutative case [MM15a]. His work shows, for instance, that there is an unique notion of a higher-order differential operator, both in the commutative and non-commutative geometry.
  [MM15a] M. Markl: *On the origin of higher braces and higher-order derivations*. To appear in Journal of Homotopy and Related Structures, DOI 10.1007/s40062-014-0079-2.

3. String field theory, quantum groups and vertex operator algebras

Two members of the team are working in this field, namely M. Markl and A. Zuevsky.

M. Markl proved that the operads describing the structure of closed and of open string field theory are terminal objects in an appropriate category. This result shows

that these fundamental objects are entirely characterized by their categorial properties. Markl's result [MM15b] is therefore of the same nature as the celebrated cobordism hypothesis.
[MM15b] M. Markl, *Modular envelopes, OSFT and nonsymmetric (non-) modular operads*. Preprint arXiv:1410.3414. Accepted for publication in J. of Noncommut. Geom.

A. Zuevsky joined the Institute only recently and most of his publications including some of those from 2014 have thus other affiliations. The 2014 publications with affiliation to our institute are: [434340], [434354], [436739].

In [434335], we compute the partition and *n*-point "intertwined" functions for modules of vertex operator superalgebras with formal parameter associated to local parameters on Riemann surfaces obtained by self-sewing of a lower genus Riemann surface. We introduce the torus intertwined n-point functions containing two intertwining operators in the supertrace. Then we define the partition and n-point correlation functions for a vertex operator superalgebra on a genus two Riemann surface formed by self-sewing of the torus. For the free fermion vertex operator superalgebra we present a closed formula for the genus two continuous orbifold partition function in terms of an infinite dimensional determinant with entries arising from the original torus Szegő kernel. We showed that this partition function is holomorphic in the sewing parameters on a given suitable domain and possess natural modular properties. We describe modularity of the generating function for all n-point correlation functions in terms of a genus two Szegő kernel determinant (see [Z14] for a review). In [434340], using combinatorial way to compute the torus two-point function for the b-c system and bosonisation formula for rank one Heisenberg vertex operator algebra, we derive a new formula for the genus one prime form.
In [434365], we identified the full Lie-algebraic structure of the Generalized Davey-Stewartson (GDS) system of equations with symmetries of a specific of continual Lie algebras. In particular, we show that they are related to two copies of the Poisson bracket continual Lie algebra. We construct in [434354] generators of the principle Heisenberg subalgebra in the quantized universal enveloping algebra $U(sl_2\hat{})$. Applications for exactly solvable models are proposed. We construct in [436739] quantum soliton solutions to the homogeneous grading case of the affine Toda models, in particular, the sine--Gordon equation.
[Z14] A. Zuevsky, *Generalized vertex algebra of intertwiners and correlation functions on self-sewn Riemann surfaces*, preprint Math. Inst. Acad. Sci. Czech Republic

4. Geometric structures on manifolds and their deformations

This topic has been studied by Hông Vân Lê. In dimensions 6, 7, 8 a classification of many interesting geometric structures is reduced to a classification of orbits in real Z_m-graded semisimple Lie algebras. In [359321], a method is proposed to classify these orbits, which extends the Vinberg method for the case over the field C to the more complicated case over the field R. With M. Munir (Abdus Salam School of Mathematics in Lahore) we classified homogeneous G_2-structures [380315]. In particular, H. V. Lê proposed a method to classify certain finite subgroups of compact Lie groups. Geometric structures which are generalizations of G_2-structures in higher dimensions are studied in [392316]. H. V. Lê proposed a cohomological method to study these structures. In collaboration with Y-G. Oh (Institute of Basic Sciences, Center for Geometry and Physics in Pohang), A. Tortorella and L. Vitagliano (Salerno University) the deformation problem of coisotropic submanifolds in Jacobi manifolds

was reduced to a deformation problem in algebraic category [LO12], [LOTV14], thus extending previous results by Oh-Park, Catanese-Felder, and Schaetz-Zambon. Using the idea proposed by H. V. Lê in her joint paper with Oh [LO12] for the study of Hamiltonian deformations, it was proved in the subsequent paper [LOTV12] that the Hamiltonian equivalence and the gauge equivalence are the same, which is important for the study of related moduli spaces.
H. V. Lê also introduced an evolution equation to find a holomorphic connection over compact Kaehler manifolds [342846] and studied the rigidity of holomorphic connections [342846].
[LO12] H. V. Lê and Y. G. Oh, *Deformations of coisotropic submanifolds in locally conformal symplectic manifolds*, accepted for publication in Asian Journal of Mathematics, arXiv:1208.3590
[LOTV14] H. V. Lê, Y. G. Oh, A. G. Tortorella and L. Vigtaliano, *Deformations of coisotropic submanifolds in abstract Jacobi manifolds*, arXiv:1410.8446

5. Information geometry and complexity

This is another topic studied by Hông Vân Lê.
Information geometry provides a way of understanding information-theoretic quantities, statistical models and corresponding statistical inference methods in geometric terms. Extending the original concept proposed by H. V. Lê in [AJLS1], the new concepts of statistical models and tensor fields on them were proposed by developing a theory of Banach spaces and smooth mappings between them. The concept encompasses all models known in information geometry. As a result, we extended the Chentsov theorem characterizing the Fisher metric, which has been proved by Chentsov in 1972 for finite sample spaces, to general infinite sample spaces [AJLS1, AJLS2, L2013] and derive a classification of invariant covariant tensors on statistical models [AJLS2]. We interpret several important statistical concepts and results, e.g. sufficient statistics [AJLS2], monotonicity theorem [L2013, AJLS2], Cramer-Rao inequality [AJLS2] using our new theory.
[AJLS1] N. Ay, J. Jost, H.V. Lê and L. Schwachhöfer, *Information geometry and sufficient statistics*, Probab. Theory Relat. Fields DOI 10.1007/s00440-014-0574-8, arXiv:1207.6736
[AJLS2] N. Ay, J. Jost, H. V. Lê and L. Schwachhöfer, *Information geometry,* book, submitted
[L2013] H.V. Lê, *The uniqueness of the Fisher metric as information metric*, arXiv:1306.1465, (submitted)

6. Symplectic spaces with singularities and cohomology theories of locally conformal symplectic manifolds

In [391461], [437490] a new algebraic concept for singular symplectic spaces has been proposed and a cohomology theory for these spaces has been developed. In [441464], the new cohomology theory for symplectic manifolds proposed by Tseng-Yau has been extended to locally conformal symplectic manifolds. In particular, H. V. Lê gave an answer to Tseng-Yau question on the relation between the new cohomology theory and the deRham cohomology theory. The papers [441464], [437490] have been published in 2015.

Research Report of the team in the period 2010–2014

Institute	Institute of Mathematics of the CAS, v. v. i.

Scientific team	Differential Equations and Theory of Integral

In the period 2010–2014, we studied qualitative properties of various differential equations and their systems, generalized differential equations, difference equations, and dynamic equations on time scales. We dealt with the theory of boundary value problems, integration theory, as well as asymptotic and oscillation theory. Particular problems of our investigation are the following:

1. Integration theory and generalized differential equations

Due to the impact of Jaroslav Kurzweil, the founder of the new and powerful notion of a summation integral, this topic has a long tradition in the Institute of Mathematics. In 2011, J. Kurzweil completed his new treatment done during the years 2008-2011 and published a monograph devoted to a quite new view of these topics (see [375634]). The emphasis is on equations having possibly continuous, but not absolutely continuous solutions. Essentially, his monograph contains only new results.

Throughout the period 2010-2014 we also studied in detail the properties of the abstract Kurzweil-Stieltjes integral and its applications. We followed the previous results delivered by the former key researcher of the Institute of Mathematics, late Štefan Schwabik. In particular, we extended his results about the existence of the integral, integration-by-parts, substitution and new convergence theorems (see [385118]). All these results were substantial for the proofs of new theorems on continuous parameter dependence of solutions to abstract generalized linear differential equations. First, we assumed that variations of the potentials of the limiting equations are uniformly bounded and the potentials converge uniformly to a limit potential (see [380374]). Later we succeeded to extend this result also to the case that these variations are bounded only with a weight and we provided an example showing that our assumptions are optimal (see [427479]). Also, a rather complicated situation when the uniform convergence is violated was solved (see [369691]). As special cases, applications to dynamic equations on time scales have been presented.

In addition, existence, uniqueness, and continuous dependence of solutions of linear measure functional differential equations with infinite delay were studied and new results were presented in [434085]. The main tool was the notion of generalized linear ordinary differential equations in a Banach space. Even for equations with a finite delay, the delivered results are stronger than those known before. Applications to functional differential equations with impulses were given, as well.

In the meantime, the first monograph on the Stieltjes integral written in Czech has been completed and printed (see [382430]). Its main goal was to include the modern theory based on Kurzweil's definition of the generalized integral. This makes it possible to considerably extend the classes of integrable functions and to simplify most of the proofs of the basic properties. Applications to functional analysis and to generalized differential equations were included, too. These lecture notes now became a basis for a new monograph prepared for the World Scientific Publishing House upon the invitation by Prof. Lee Peng Yee. In the new monograph we will work mostly in the Banach space setting. In the frames of the works on it, we already succeeded to collect the available knowledge of functions with bounded semivariations into (more or less) survey paper accepted for Real Analysis Exchange. Furthermore, we are proud of our success in proving the so called "bounded convergence theorem" for abstract Kurzweil-Stieltjes integrals. To this aim, we introduced the notion of Stieltjes integral over elementary sets whose study can still bring some fruitful results.

2. Boundary value problems for ordinary differential equations

We continued the study of the questions about existence and uniqueness of solutions to various boundary value problems for non-autonomous ordinary differential equations both in the regular and singular cases. In particular, we have found new sufficient conditions for the existence and uniqueness of a periodic solution to both linear and non-linear higher-order non-autonomous ordinary differential equations, which substantially extend the results available in the existing literature (see [381249]).

For systems of half-linear differential equations subjected to non-local boundary conditions, Fredholm type theorems were proved guaranteeing their solvability provided a certain family of related half-linear problems has only the trivial solution. Using these general theorems, we derived new effective solvability criteria for the problems studied, as well as for some of their particular cases (see [362862]). It was also justified by counterexamples that the conditions obtained are non-improvable.

Moreover, new effective conditions for the existence of a maximum or anti-maximum principle of a general second-order operator with periodic conditions, as well as conditions for non-resonance, were established and compared with the related literature (see [359866]).

The Dirichlet problem for second-order ordinary differential equations is studied by many authors and investigated in detail. However, in almost all those studies, only the case is considered, when the corresponding homogeneous linear problem has only a trivial solution. The case, when the corresponding homogeneous linear problem has also a nontrivial solution, is still little investigated. In the majority of articles, the case is studied, where the first coefficient of the corresponding homogeneous linear equation is a constant and furthermore, in the simple case only, when this constant is the first eigenvalue of the homogeneous linear problem. We developed a technique which enabled us to establish efficient conditions for the solvability of the mentioned problem in the case, where the first coefficient of the homogeneous linear equation is a Lebesgue integrable function (not necessarily constant) under the assumption that solutions of the corresponding homogeneous problem have arbitrary number of zeros in the defined area (if, in homogeneous linear equation, the first coefficient is a constant, then this constant is not necessarily the first eigenvalue of the corresponding homogeneous linear problem). Theorems proved in this study not only significantly generalize and improve other authors' results, but rather cover the cases that had not been practically studied by the time the article was issued (see [387096]).

Singular non-linear boundary value problems are in the focus of our attention since the 1980's. Several members of the team became to be respected for their contribution to this topic. M. Tvrdý is a co-author of the related monograph published in 2009. During the period 2010-2014 we continued our research. In particular, new non-improvable criteria were proved for the existence as well as uniqueness of a solution to a multi-point boundary value problem for non-linear singular differential equations of the second order (see [422624]). We further mention that new effective criteria were established guaranteeing the existence of a solution to the Dirichlet problem for the second-order differential equation with singularities both in the time and the phase variables (see [397179]).

Moreover, we succeeded to extend the anti-maximum principle (called also inverse positivity) to the quasi-linear periodic problem with a sign-changing potential. Consequently, we were able to present new existence results for singular non-linear periodic problems with a quasi-linear differential operator (see [343853]). Furthermore, sufficient conditions ensuring existence and asymptotic stability of periodic positive solutions for a pipe/tank flow configuration were obtained (see [427046]). The model is a non-linear ordinary second-order differential equation with a singularity containing the second power of the derivative of the unknown function and with friction.

We also developed a method which allows us to construct upper and lower functions to certain classes of differential equations with singularities in the phase variable and to resolve thus the question on the existence of a periodic solution to a family of equations often used in physics to model natural processes (see [340552, 370679, 380708, 397163, 425959]). Among the equations to which our results are applicable belongs the equation with attractive/repulsive singularity where the singular term can be regarded as a generalized Lennard-Jones force or van der Waals attraction/repulsion force and it is widely used in molecular dynamics to model the interaction between atomic particles. In a different physical context, a periodic solution to such types of equations is equivalent to a matter-wave breather in a Bose-Einstein condensate with a periodic control of the scattering length (the mathematical model is a non-linear Schrödinger equation with a cubic term, then the method of moments leads to the study of a particular case of the above-mentioned equation). A third different range of applicability is the evolution of optical pulses in dispersion-managed fiber communication devices. Finally, the results can be applied to the Rayleigh-Plesset type equations often used in physics of fluids to model the oscillations of the radius of a spherical bubble immersed in a fluid under the influence of a periodic acoustic field.

We took advantage of the techniques developed for the second-order ordinary differential equations to open a new path in the study of DM-solitons in optical fibers. The main technical difficulty is that in the general case with arbitrary coefficients, the system cannot be rewritten as a second-order ordinary differential equation. This has forced us to develop a specific upper and lower function method for this framework (see [348283]). It is shown that such method is a natural extension to the first-order system of known results for the second-order scalar ordinary differential equations, so in this sense from a mathematical point of view it is interesting by itself.

On the other hand, we presented a formal geometric derivation of a non-equilibrium growth model that takes the form of a parabolic partial differential equation. Subsequently, we study its stationary radial solutions by means of variational techniques. It turns out that the existence or not of such solutions depends on the size of a parameter that plays the role of velocity at which mass is introduced into the system. For small values of this parameter, we then proved existence and multiplicity of solutions to a certain boundary value problem. For large values of the same parameter, we proved the nonexistence of solutions. We also provided rigorous bounds for the values of this parameter, which separate existence from nonexistence (see [393489, 423454]).

At last, we also investigated equations with the so-called **strong singularities**. As it is well known, the boundary value problems for strongly singular linear differential equations do not always have the Fredholm property. For the second-order linear differential equations with strong singularities, the Dirichlet problem was studied. We have established counterparts of Fredholm's theorems on the solvability of this problem that cover, for instance, the case of the Bessel equation in which strongly singular coefficients are present (see [430385, 430387]).

3. Boundary value problems for functional differential equations

We collected our previous results concerning the questions of existence, uniqueness, and non-negativity of a solution to the initial value problem for two-dimensional systems of functional differential equations and published them as a long research paper (127 pp., see [347003]). Moreover, a long-term work concerning the existence of bounded solutions to first-order functional differential equations was finished. As a result, a long research paper was published (104 pp., see [423453]) containing non-improvable conditions guaranteeing the existence, uniqueness, and sign properties of a bounded solution to a non-linear first-order functional differential equations. The main importance of the results obtained consists in the fact that the operator on the right-hand side is, in general, of non-Volterra type.

Furthermore, we continued in our previous studies of various topics of the qualitative theory of functional differential equations and their systems. In particular, efficient conditions guaranteeing that the solution set of a first-order linear functional differential equation is one-dimensional, generated by a positive monotone function, were established. The results obtained were applied to get new conditions sufficient for the solvability of a class of boundary value problems for first-order linear functional differential equations (see [384217]). Further, new optimal criteria were derived for the existence and uniqueness of a periodic solution to systems of functional differential equations in both linear and non-linear cases (see [351072, 373148, 351074]). Moreover, Fredholm-type theorems for boundary value problems for systems of non-linear functional differential equations were established (see [430349]). The theorem generalizes results known for the systems with linear or homogeneous operators to the case of systems with positively homogeneous operators. We also proved new effective conditions guaranteeing the unique solvability of a two-point boundary value problem for the third-order linear functional differential equations (see [380320, 389752]). A great importance was put on the question, whether there exist solutions which are positive inside the considered interval.

It is known that the condition of non-negativity of non-diagonal coefficients in the continuous matrix function P is necessary and sufficient for the non-negativity of all components of the solution vector to a system of linear differential inequalities with the matrix P. Although this result was extended to various boundary value problems and to delay differential systems, analogs of these strong restrictions on non-diagonal coefficients persist in all assertions of this kind. We proposed a certain method to compare only one component of the solution vector, which does not require such heavy restrictions. The main idea of our approach is to construct a first-order functional differential equation for the *n*-th component of the solution vector and then to use assertions, obtained recently for first-order scalar functional differential equations with general operators on the right-hand sides (see [378921]). This indicates the importance of investigations of scalar equations written in a general operator form, where only properties of the operators independent of their explicit forms are assumed.

Finally, for a wide class of nonlocal boundary value problems, we studied conditions of a general character ensuring the existence and uniqueness of a solution, which contain various results based on the use of differential inequalities and unify certain ideas used in the proofs of various previously known statements (see [343938]). The case of measure differential equations in the space of functions of bounded variation was also considered from this viewpoint (see [397182]).

Let us mention also that the solvability of initial value problems for functional differential equations with **singular** coefficients was studied and certain sufficient conditions were obtained (see [346706, 374152]).

At last, for **strongly singular** higher-order linear differential equations with deviating arguments subjected to two-point conjugated, right-focal, and non-local boundary conditions, Agarwal-Kiguradze type theorems were established, which guarantee the presence of Fredholm's property for the above mentioned problems. We also provided easily verifiable best possible conditions guaranteeing the existence of a unique solution of the studied problems (see [380543, 395521, 423451]). On this basis, for strongly singular higher-order non-linear functional differential equations subjected to the above-mentioned boundary value conditions, several solvability conditions were proved by the a priori boundedness principle (see [391446, 423450]).

As for **partial** differential equations, we followed our previous results concerning the Carathéodory solutions to partial functional-differential equations of **hyperbolic type** (see [427045]). We obtained new results about existence, uniqueness, and continuous dependence on parameters of the Carathéodory solutions to the Cauchy problem for linear partial functional-differential equations of hyperbolic type. A theorem on the Fredholm alternative was also proved (see [377958]). The results obtained are new even in the case of equations without argument deviations, because we do not suppose absolute continuity of the function the Cauchy problem is prescribed on, which is a rather usual assumption in the existing literature. We studied these topics also for the "full" second-order linear hyperbolic equations, where partial derivatives of the first order are involved in the equation. It needs, in particular, to find suitable function spaces serving as domains of operators on the right-hand side of the equation and to prove new compactness results in these spaces. We succeeded to find the effective sufficient conditions for the existence, uniqueness, and non-negativity of the Carathéodory solutions to the Darboux problem for "full" linear hyperbolic equation in a particular case, when the images of the operators acting on right-hand side of the equation are in the spaces of essentially bounded measurable functions (see [362930]).

4. Asymptotic properties of solutions

We investigated a cyclic **system** of *n* differential equations of Emden-Fowler type, which includes several important objects, like *n*-th order Emden-Fowler differential equations or second-order systems which arise out when studying radial solutions of partial differential systems with *p*-Laplacian operators. The theory of regular variation played an important role in our considerations. We showed that extreme solutions (strongly decreasing and strongly increasing ones) are regularly varying. What makes our result to be particular is that we dealt with ALL solutions in a given basic asymptotic class. Moreover, we established asymptotic formulae for such solutions. The method which was developed is believed to have a potential for examination of some other type solutions or equations (see [385126, 429349]).

We also established conditions guaranteeing that monotone solutions of **half-linear** equations belong to the class Gamma (in the de Haan sense), see [429354]. As by-products, we get their rapid variation and a half-linear extension of classical Hartman's result. We dealt with solutions in the class Pi of the so-called nearly linear equations in which nonlinearities are in the form of regularly varying functions of index 1; such equations may contain some important cases of a generalized Laplacian (see [429228]).

5. Oscillation theory

We developed a technique which enabled us to find new comparison theorems and oscillation criteria for half-linear **dynamic** equations on an arbitrary time scale. In the comparison results, we showed that (non)oscillation of the equation is maintained when the power in the nonlinearity is somehow changed (see [389740]). The most interesting feature of this statement is the fact that there is a considerable dependence on the graininess of the time scale. Further, sharp conditions which guarantee oscillation resp. non-oscillation of all nontrivial solutions to the equation are established (see [349242, 343477]). For historical reasons, we call them criteria of Hille-Nehari type. Oscillation theory, including oscillation of dynamic equations on time scales, is still very active. Many results have appeared in the recent years, but they usually have some restrictions, typically of two types: The constants in the criteria are not optimal and/or the choice of time scales is strictly limited. This undesirable fact disables examination of many important cases. Our results substantially remove these restrictions. Moreover, they reveal that the value of the (critical) constant, which is in principle the best possible, depends on the graininess of time scales and on one of the equation coefficient.

One of the well-known oscillation results for linear second-order **ordinary** differential equations is the Kamenev criterion, which requires that the upper bound of a certain expression is equal to plus infinity. In the literature, numerous generalizations of this criterion can be found for various differential equations and their systems. However, there are only few oscillation results complementing the Kamenev criterion, i.e., results which are applicable in the case when the upper bound of the Kamenev type expression is finite or minus infinity. We found a certain counterpart of the Kamenev criterion for linear second-order ordinary differential equations and we showed that the result obtained allows one to derive a generalization of the well-known Hartman-Wintner theorem covering also the cases, when the classical Hartman-Wintner theorem cannot be applied (see [367503]).

Using our previous experience with the Riccati equation corresponding to non-oscillatory ordinary differential equation, we obtained new Hille and Nehari type oscillation criteria for the second-order **delay** differential equations (see [391352, 359865]). The main difficulty which had to be overcome in this study is to find suitable upper and lower a priori estimates of non-oscillatory solutions to a delay equation. The oscillation results obtained for delay equations are as sharp as precise a priori estimates we are able to prove.

We also found new Wintner, Hille, and Nehari type oscillation and non-oscillation criteria for two-dimensional linear **systems** with "anti-diagonal" matrices in the case, when one of the coefficients preserves its sign and it is integrable on the half-line (see [395469]).

Finally, we worked on the discrete Karamata theory, and applied it in the study of asymptotic properties of linear and half-linear **difference** equations (see [351149]). We introduced the concept of regular variation on time scales, in order to examine asymptotics of dynamic equations (see [333009]). We studied in particular two cases: 1) The graininess of a time scale is "small" (namely $o(t)$). Then we get the continuous-like theory. 2) The graininess is linear (as in the quantum calculus). Then we have a somehow natural setting for regular variation, and many properties are less complicated. We established sufficient and necessary conditions guaranteeing regular variation of solutions of linear **dynamic** equations (with small graininess) and of linear ***q*-difference** equations (see [374109, 372060]). We revealed, for example, that while in the former case, the conditions must be in an integral form, the latter case requires a substantially simplified form.

6. Constructive methods for boundary value problems

One of the topics under investigation concerns a constructive, in a certain sense, approach to boundary value problems, which are based upon reductions of Lyapunov-Schmidt type and allows one to both study the solvability and construct approximations to the solution in question. This group of methods is particularly interesting due to the fact that, in this context, results of computation can be efficiently used in the rigorous analysis of solvability. In this way, one can start the analysis of a concrete problem directly by constructing approximations and only then continue to establishing the existence of a solution based on the information gathered during the computation.

We developed new techniques of this kind suitable for use for various boundary value problems for ordinary differential systems (in particular, periodic type problems, Dirichlet, more general two-point, three-point, and certain classes of non-local problems). The main assumption on the equation in the corresponding results is that the non-linearity is Lipschitzian on a compact set. Neither the global Lipschitz condition is needed, nor is the unique solvability of the initial value problem for the given differential system assumed. Suitable smallness conditions on the non-linearity then guaranteed the convergence of iterations and applicability of the scheme (see [370532, 392479]).

We studied the possibility to overcome the limitation consisting in the smallness conditions mentioned. For this purpose, as it turned out, we were able to introduce a natural idea based on the interval halving technique, by using which one can always modify the scheme so that its applicability is always guaranteed regardless that smallness assumption (see [392473, 392466, 436533]). The "price" to be paid is the increase of dimension of a finite-dimensional determining system which should be solved. This works for a wide range of boundary value problems (in contrast, e.g., to the convergent, under similar assumptions, modification of the known Cesari method). Existence theorems based on properties of approximations are proved and a natural relation to the known continuation theorems is described. Certain classes of functional differential systems were studied also in this way (see [362702, 376645, 377478, 392460]). Note that the theorems of this kind have, as a rule, a constructive character, because the approximations are constructed in a closed form and assumptions are always formulated in terms of expressions that can be computed explicitly. We studied the use of the schemes for various concrete examples and discussed its possible modifications which are most suitable for practical computations (including producing a code for systems of symbolic computations, e.g., Maple). Computer-oriented verification of assumptions was also addressed.

7. Theory of functions

Theory of generalized linear differential equations was also utilized to introduce new definitions of the exponential, hyperbolic and trigonometric functions (see [421706]). Some basic properties of these generalized functions were described and it was shown that the time scale elementary functions with Lebesgue integrable arguments represent are special cases of those generalized elementary functions introduced by us.

The notion of absolute continuity of functions of a single variable has been generalized to more variables in many ways (Schwartz, Banach, Tonelli, Malý, Carathéodory, etc.). Each of these constructions observes only some of properties known in one dimensional case like continuity, differentiability almost everywhere or integration by parts, the others are naturally lost. If the second-order hyperbolic differential equations with the "integrable" right-hand sides are considered, "strong" solutions are usually defined as functions of two variables which are absolutely continuous in the sense of Carathéodory. The definition of an absolutely continuous function of two variables was introduced by Carathéodory in 1918 and it is known that the set of those functions coincides with the class of functions admitting a certain integral representation. We proved in detail that absolutely continuous functions in the sense of Carathéodory can be also equivalently characterized in terms of their properties with respect to each of variables (see [349672]). Moreover, in our studies of hyperbolic equations, we needed to show that a certain function of two variables is absolutely continuous. To prove it, we were forced to discuss the absolute continuity of certain composite functions and to prove new theorems on differentiation of a Lebesgue integral with respect to parameters (see [379720]).

Research Report of the team in the period 2010–2014

Institute	Institute of Mathematics of the CAS, v. v. i.

Scientific team	Numerical Analysis

1. General overview

The research of the Numerical Analysis team during the period 2010–2014 was concentrated mainly on various aspects of the finite element method which is at present supposed to be the most often used tool for numerical solution of partial differential equations. In particular, we focused on generation and refinement of finite element meshes, on a priori and a posteriori error estimates, on domain decomposition methods and parallel computing, on discrete maximum principles, and on various applications of mathematics and numerical methods especially in astronomy, biology, and engineering. A detailed description of the research and results obtained within these topics follow in the next sections.

In years 2010–2014 the team members (Křížek, Kůs, Segeth, Šístek, and Vejchodský) published 35 papers in impact factor journals and more than 10 papers in journals without impact factor included in the Mathematical Reviews. They also published a number of contributions to proceedings of both national and international conferences. These publications were obtained in cooperation with more than 25 academic institutions from USA, Canada, Chile, China, Finland, Great Britain, Germany, Mexico, Spain, the Netherlands, and the Czech Republic.

Team members gained several notable achievements during the years 2010–2014. In 2012 Michal Křížek was awarded the Medal of the University of Jyväskylä (Finland) for a long-term collaboration with Prof. Pekka Neittaanmäki with whom he wrote 25 research papers, two monographs, and edited four conference proceedings. Michal Křížek also got the Josef Hlávka Award for Scientific Literature (2010), the Bernard Bolzano Medal for Merits in Mathematical Sciences (2012), and the Prize for Promotion and Popularization of Research, Experimental Development and Innovations of the Academy of Sciences of the Czech Republic (2013). Karel Segeth received in 2013 the Honorary Medal for Mathematics awarded by the Czech Mathematical Society. Jakub Šístek won the Otto Wichterle Prize in 2013 and gained as a CO-PI grant by the Czech Science Foundation (Advanced Methods for Flow-field Analysis, 2014-2016). Tomáš Vejchodský became Associated Professor at the Faculty of Mathematics and Physics of Charles University in 2012. In 2013 he gained the Intra-European Marie Curie Fellowship funded by the European Commission.

The team members organized two international conferences Application of Mathematics (2012 and 2013) and three national conferences entitled Programs and Algorithms of Numerical Mathematics (2010, 2012, 2014). For each of these conferences they published a book of proceedings.

Below, we introduce more detailed overview of the team members and in the subsequent sections we present the main research achievements of the team during the years 2010–2014, where we refer to the literature using the ASEP identifiers.

2. Team members

Michal Křížek is a senior researcher at the Institute of Mathematics AS CR and the Head of the Department of Constructive Methods of Mathematical Analysis. His research interests during 2010–2014 included numerical analysis, discrete geometry, number theory, graph theory, and mathematical methods in astronomy, cosmology, entomology, and theoretical biology. He has a wide network of international collaborators. For example Jan Brandts (University of Amsterdam), Wei Chen (Shandong University, China), Antti Hannukainen (Helsinki University of Technology), Sergey Korotov (Basque Center for Applied Mathematics), Liu Liping (Lakehead University, Canada), Florian Luca (Universidad Nacional Autonoma de Mexico), Pekka Neittaanmäki (University of Jyväskylä), Hans-Goerg Roos (Technical University Dresden), Igor Shparlinski (Macquarie University, Australia), and Lawrence Somer (Catholic University of America, Washington, D.C.).

Pavel Kůs was the team member as a PhD student until 2011, when he defended his PhD thesis under the supervision of Tomáš Vejchodský. Then he worked at the Institute of Thermomechanics of the AS CR and at the University of West Bohemia. In April 2015 he joined our team again, this time on a post-doc position.

Karel Segeth is a senior researcher in the Institute of Mathematics AS CR. Since 2010 he works part time. His research interests include a posteriori error estimators, adaptive methods, and smooth interpolation and approximation of functions. He collaborates with Prof. Ivo Babuška (University of Texas at Austin, USA), Jan Hrabě (State University of New York, USA), and Josef Ježek (Faculty of Science, Charles University in Prague).

Jakub Šístek joined the Institute of Mathematics AS CR in 2009. He held a post-doc position since January 2009 until December 2012, and since January 2013, he has been a Research Fellow of the Institute. For the whole time, his main interest has been algorithms for solving large-scale systems of linear equations arising from the finite element method (FEM), and domain decomposition (DD) techniques in particular. In addition, he has been interested in parallel programming, since DD methods are among the most efficient mathematical algorithms for solving systems of equations on massively parallel computers. Jakub Šístek has collaborated closely with Jan Mandel (University of Colorado Denver) and Bedřich Sousedík (University of Colorado Denver, now University of Maryland Baltimore County), who are leading experts in DD methods. In addition, Jakub Šístek has maintained close collaboration with Pavel Burda, Marta Čertíková and Jaroslav Novotný from the Czech Technical University in Prague. With respect to large-scale applications to problems of incompressible flows related to insect flight, Jakub Šístek has also initiated collaboration with Fehmi Cirak (University of Cambridge), with whom he has spent 7 months as a Research Associate in 2011.

Tomáš Vejchodský works as a researcher in the Institute of Mathematics AS CR since 2003. Since March 2013 he spent 18 moths at the University of Oxford as the Marie Curie Fellow. He specializes on discrete maximum principles for various types of finite elements including higher-order approximations, on guaranteed a posteriori error estimates for finite element solutions of elliptic problems and guaranteed two-sided bounds of eigenvalues of elliptic differential operators, and on applications of the mathematical analysis and numerical methods to biological problems including formation of spatial patterns and stochastic modelling of biochemical processes. He collaborates with Sergey Krotov (Basque Center for Applied Mathematics, Spain) and

Antti Hannukainen (Helsinki University of Technology, Finland), Mark Ainsworth (Brown University, Providence, USA), Ivana Šebestová (University of Concepcion, Chile), Ramon Grima (University of Edinburgh), Václav Klika (Czech Technical University, Prague), Milan Kučera and Martin Vaeth (Institute of Mathematics AS CR, Prague), and with several researchers from the mathematical biology group of the University of Oxford (Philip K. Maini, Radek Erban, Shuohao Liao, Andrew Duncan, and others).

3. Generation and refinement of finite element meshes

We examined various types of finite element meshes and succeeded to prove that the maximum angle condition is not necessary for the convergence of the finite element method for elliptic boundary value problems. Numerical tests were performed at Aalto University (Finland) and theoretical part of the paper was done in collaboration with the Basque Center for Applied Mathematics (Spain). This result was published in Numerische Mathematik [371026].

In collaboration with Basque Center for Applied Mathematics (Spain) and Helsinki University of Technology (Finland), we investigated the bisection algorithm for refining finite element meshes that is often used to refine tetrahedral partitions. In [346707], we generalized conforming bisection algorithm that allows both global and local nested refinements of triangulations without generating hanging nodes. Some regularity properties of refined triangulations were proved. It is also shown how to modify the proposed algorithm in order to generate anisotropic meshes with high aspect ratio. Next, we studied numerical regularity of the face-to-face longest-edge bisection algorithm in [428478]. We developed an algorithm for local tetrahedral refinements around edges of polyhedral domains, where singularities of solutions of PDEs may appear [393009], and we have shown how to generate and conformally refine nonobtuse tetrahedral meshes locally in the neighbourhood of a polygonal face or a polygonal interior interface of a three-dimensional polyhedral domain. Such partitions guarantee the validity of discrete maximum principle for the Poisson equation [358144].

We have been also interested in geometric restrictions of refinements in higher dimensions, an important topic in solving higher-dimensional PDEs. This research was performed together with the Basque Center for Applied Mathematics (Spain) and the Korteweg-de Vries Institute of the University of Amsterdam (the Netherlands). We succeeded to prove that for the dimension greater than three there does not exist any simplex that could be partitioned into subsimplices that are congruent to the original simplex [430376]. We also proved that there are only two nonobtuse binary triangulations of the unit n-cube [387003].

We also focused on a construction of conforming basis functions of polynomial order up to 10 for elliptic and electromagnetic problems on 3D hexahedral meshes with no restrictions on the level of hanging nodes. Using this construction, it was possible to perform automatic hp-adaptivity in 3D with no extra refinements, whose sole purpose would be ensuring some kind of regularity of the mesh. This research was summarized in the doctoral theses of Pavel Kůs supervised by Tomáš Vejchodský. The thesis was entitled "Automatic hp-Adaptivity on Meshes with Arbitrary Level Hanging Nodes in 3D" and it was defended in 2011. It comprises description of algorithms and numerical simulations verifying expected convergence properties of the method. Special care has been devoted to efficient integration on hexahedral elements of higher order. All algorithms have been implemented in an open-source

C++ library Hermes3D, which has been developed in collaboration with prof. Pavel Šolín, formerly from the University of Texas at El Paso. Certain results of this thesis were also published as research papers.

In [391351] we proposed a novel approach for mesh adaptation for the solution of the chemical Fokker-Planck equation. This equation describes the probability of finding the stochastic model of a chemical system in a given state. We use short stochastic simulations to identify regions where the stationary distribution of the probability concentrates and adapt the finite element mesh accordingly. Provided numerical examples demonstrate that the proposed method is competitive with the existing adaptive methods based on the traditional a posteriori error estimators.

4. A priori and a posteriori error estimates

We derived two-sided a priori bounds on the finite element discretization error of elliptic boundary value problems by means of the interpolation error. The theoretical part was achieved in collaboration with Technical University Dresden (Germany) and numerical tests were performed at Shandong University (China). This result was published in [359285]. We also succeed to generalize the well-known Zlámal condition on the minimum angle of triangular finite elements into an arbitrary space dimension. This result has been obtained in collaboration with the University of Amsterdam (Netherlands) and the Basque Center for Applied Mathematics (Spain) and we published it in [360530].

We examined pyramidal elements that are used to connect tetrahedral and hexahedral elements in the finite element method. Numerical results were obtained in collaboration with Department of Mathematical Sciences of Lakehead University (Canada), Dongguan University of Technology (China), and Institute of Computer Science of Hunan Normal University (China). We developed and presented a new symmetric composite pyramidal finite element which yields a better convergence than the nonsymmetric one. It has 14 degrees of freedom and its basis functions are incomplete triquadratic polynomials. The space of ansatz functions contains all quadratic functions on each of four subtetrahedra that form a given pyramidal element. We published this result in [355509]. In addition, we derived three new higher order numerical cubature formulae for pyramidal elements. This research resulted in paper [392337].

Further, we have been interested in the critical comparison of various a posteriori estimates of the error of the finite element solution of elliptic problems of second as well as fourth order. Unfortunately, these estimates often contain constants that are not computable, see e.g. [353075]. Estimates without these constants are presented in journal papers [367487], [379721], [391349], [425594], [431616], and one proceedings contribution for both the partial differential equations and eigenvalue problems. We were inspired by a certain earlier work, where an upper error bound is obtained by solving local Neumann problems and hence the resulting error estimator is not fully computable. In [367487] we remove this limitation and construct the first fully computable, robust upper bounds in the setting of singularly perturbed problems discretised by the finite element method. In [431616] we generalized this result in three aspects: (i) arbitrary dimension, (ii) mixed boundary conditions, and (iii) non-constant reaction coefficient. This generalization was nontrivial, because we had to construct new local flux reconstruction and kept the robustness of the newly proposed fully computable and guaranteed upper bound on the error. We obtain these two results in collaboration with Mark Ainsworth (Brown University, USA).

Paper [379721] presents a theory of complementary error estimates for approximate solutions of elliptic reaction-diffusion problems obtained by arbitrary conforming numerical method. These estimates naturally provide guaranteed upper bounds on the energy norm of the error. The paper provides sufficient and necessary conditions for the efficiency and asymptotic exactness of these bounds. It also presents the hypercircle approach, which enables to construct a special approximation and the error of this approximation can be computed exactly. Thus, the resulting error estimate is actually not an estimate, but exact formula for the size of the error. The paper is concluded by numerical experiments showing the performance of the method.

We also derived guaranteed upper bounds of the energy norm of the approximation error for systems of linear elliptic partial differential equations. This generalizes the complementarity error estimates known for scalar elliptic problems to general diffusion-convection-reaction linear elliptic systems. For systems we prove analogous properties as for the scalar case. We apply this general theory to linear elasticity as well as to chemical systems with reactions of at most first order. The sharpness of the obtained upper bounds and their behaviour in the adaptive procedure was tested numerically. These results were published in [391349].

In [425594] we address the especially difficult problem of computing lower bounds of eigenvalues of elliptic differential operators. In general, we present a numerical scheme for computing guaranteed two-sided bounds for principal eigenvalues of symmetric linear elliptic differential operators. The approach is based on the Galerkin method, on the method of a priori–a posteriori inequalities, and on a complementarity technique. The two-sided bounds are formulated in a general Hilbert space setting and as a by-product we prove an abstract inequality of Friedrichs-Poincaré type. As an application, we use the computed lower bound on the spectrum to find a guaranteed upper bound on the optimal constant in the Friedrichs, Poincaré, and trace inequalities. We illustrate the accuracy of the method on several numerical examples. These results were obtained in collaboration with I. Šebestová from the University of Concepcion, Chile.

We also worked on a nodal $O(h^4)$-superconvergence by averaging piecewise linear, bilinear and trilinear finite element approximations. Numerical tests were performed at Helsinki University of Technology (Finland), whereas the theoretical part was done in cooperation with the Basque center of Applied Mathematics (Spain). This topic leads to the publication [338973].

5. Domain decomposition methods and parallel computing

The team has contributed to the field of domain decomposition (DD) methods and high performance computing. Results in this field were obtained in collaboration with Jan Mandel (University of Colorado Denver), Bedřich Sousedík (University of Colorado Denver, now University of Maryland Baltimore County), and the group from the Czech Technical University in Prague (Pavel Burda, Marta Čertíková and Jaroslav Novotný).

Perhaps the most important result in this field comprises an application of the Balancing Domain Decomposition based on Constraints (BDDC) method to saddle-point systems arising from the FEM solution of the Stokes problem [359432]. This paper presents a pioneering application of the BDDC method to the problem of Stokes flow. The unique feature of the paper is that the finite element method with continuous approximation of pressure was used for the first time in connection to BDDC. Numerical results obtained by a parallel implementation of the algorithm suggested the optimality of BDDC also for this important case.

An overarching aim during the period 2010-2014 was an extension of the BDDC algorithm towards increased robustness for complicated problems and increased scalability of a multilevel method, including massively parallel implementations. A combination of BDDC with the widely used frontal algorithm for direct solution of systems of linear equations was developed and published in [342831]. In BDDC as well as in FETI-DP, the two most recent iterative substructuring methods, the definition of the coarse space relies on a 'minimal' set of coarse nodes (sometimes called corners) which assures invertibility of local subdomain problems and also of the global coarse problem. This basic set is typically enhanced by enforcing continuity of functions at some generalized degrees of freedom, such as average values on edges or faces of subdomains. In [379744], existing algorithms for selection of corners were revisited, and a new heuristic algorithm was proposed. Considering faces as the basic building blocks of the interface, inherent parallelism and better robustness with respect to disconnected subdomains were among positive features of the new technique. Another intermediary step was the development of an automatic construction of the coarse space adjusted to the system of equations. The Adaptive BDDC method was extended to the selection of face constraints in three dimensions in [379792]. Finally, to enhance parallel scalability of the BDDC method, the Adaptive BDDC was combined with the Multilevel BDDC method in [423788]. The developed algorithm of the Adaptive-Multilevel BDDC method allows an adaptive selection of constraints on each decomposition level. The effectiveness of the method was illustrated on several engineering problems of linear elasticity.

The developed parallel implementation of the resulting Adaptive-Multilevel BDDC method was released as an open-source library BDDCML [358106] in 2011. The solver shows good scalability as well as applicability to very large problems (tested up to 2 billion unknowns) and core counts (up to 65 thousand). The solver is already interfaced by several external codes for the task of solving systems of linear equations arising from finite and spectral element methods.

Apart from the main research focus on DD methods, evolutionary algorithms for multiobjective optimization were also studied in collaboration with the Aeronautical Test and Research Institute in Prague. Paper [346694] introduced a new mechanism for selecting individuals to the so-called Pareto archive. It was combined with a micro-genetic algorithm and tested on several problems. The ability of the approach to produce individuals uniformly distributed along the Pareto set without negative impact on convergence was demonstrated. The new concept was confronted with the established NSGA-II, SPEA2, and IBEA algorithms from the PISA package. Another studied effect was the size of population versus number of generations, with an emphasis on very small populations.

6. Discrete maximum principles

Within this topic we published journal papers [352122], [352123], [370385] and three contributions in proceedings of international conferences. We value most the proof of the discrete maximum principle for one-dimensional reaction-diffusion problems discretised by higher-order finite element method [352122]. It was an extremely difficult problem to prove and the crucial condition, namely the nonnegativity of the corresponding discrete Green's function, was not possible to verify analytically.

Therefore, it was verified using the interval arithmetic for polynomial approximations of degrees up to 10.

In paper [352123] we analysed the discrete maximum principle for a three-dimensional parabolic problem discretised by prismatic finite elements and we found a geometric condition guaranteeing the discrete maximum principle. Roughly speaking, the angles of the triangular bases of the prisms cannot be too small nor too large and the altitude of the prism cannot be too high. In addition, if the original problem contains nonzero reaction term then sizes of all prisms have to be sufficiently small.

We also published the survey paper [370385]. This paper provides equivalent characterization of the discrete maximum principle for Galerkin solutions of general linear elliptic problems. The characterization is formulated in terms of the discrete Green's function and the elliptic projection of the boundary data. This general concept is applied to the analysis of the discrete maximum principle for the higher-order finite elements in one-dimension and to the lowest-order finite elements on arbitrarily dimensional simplices. This paper surveys the state of the art in the field of the discrete maximum principles and provides new generalizations of several results.

The proceedings contributions contained results that we could not prove analytically due to their complexity. For example, we present there numerical analysis the discrete maximum principle for the Poisson equation discretised by higher-order triangular elements. This analysis showed that the discrete maximum principle in its strong form cannot be valid for higher-order approximations, but it also reveals that a certain minimum angle condition for the validity of a weaker form of the discrete maximum principle can exist even for higher-order approximations. Further, we compared the simplicial and block finite elements especially with respect to the sufficient geometric condition for the validity of the discrete maximum principle. Finally, we proved necessary and sufficient condition for the validity of the discrete maximum principle for a general reaction-convection-diffusion problem in one-dimension discretised by the standard finite element method. This is an exceptional result, because all other known conditions for the discrete maximum principle are sufficient only.

7. Applications of mathematics and numerical methods

Recently, our team also became interested in applications of mathematics in fluid mechanics, namely in the problem of *identification of vortices in flow fields*. This research stems from the collaboration with Václav Kolář (Institute of Hydrodynamics of AS CR) and it has resulted in a project funded by the Czech Science Foundation (Advanced Methods for Flow-field Analysis, 2014-2016) and several publications, including [398283]. In this reference, a new kinematic quantity measuring the average corotation of material line segments near a point was introduced and applied to vortex identification. At a given point, the vector of average corotation of line segments is defined as the average of the instantaneous local rigid-body rotation over `all planar cross sections' passing through the examined point.

We have been interested in *applications of mathematical and numerical methods in astronomy.* We have shown that the Solar system slightly and continually expands. In particular, we show that Earth, Mars, and the other planets were closer to the Sun 4.5 Gyr ago. The recession speed of about 5 m/yr of the Earth from the Sun is comparable with the Hubble constant and it seems to be just right for an almost constant influx of the solar energy from the origin of life on Earth up to present over which time the Sun's luminosity has increased approximately linearly. This represents an important support for the so-called weak form of the Anthropic Principle. Moreover, this example shows that the law of energy conservation is slightly violated in reality, whereas the Newtonian and Einstein theory of gravity are formulated in such a way that this law holds. This research resulted in publication [360620] authored by Michal Křížek.

In [434759] we criticize the standard cosmological model that claims that 27 % of the Universe consists of some mysterious dark matter, 68 % consists of even more mysterious dark energy, whereas only less than 5 % corresponds to the usual (baryonic) matter. We show that the proposed ratio 27:5 between the amount of dark matter and baryonic matter is considerably overestimated. Dark matter and partly also dark energy might result from inordinate extrapolations, since reality is identified with its mathematical model which is described by the Friedmann differential equation. Especially, we should not apply results that were verified on the scale of the Solar system during several hundred of years to the whole Universe and extremely long time intervals without any bound on the modelling error. This result has been obtained together with the Catholic University of America, Washington, D.C.

Another field of applications we are interested in is *biology and biochemistry*. We examine the standard genetic code with three stop codons UAA, UAG and UGA. Assuming that the synchronization period of length 3 in DNA or RNA is violated during the transcription or translation processes, the probability of reading a frameshifted stop codon is higher than if the code would have only one stop codon. Consequently, the synthesis of RNA or proteins will soon terminate. In this way, cells do not produce undesirable proteins and essentially save energy. This hypothesis was numerically tested on Drosophila genome, where the detection of frameshifted stop codons is even higher than the theoretical value. These tests illustrate that C –> U and C –> T mutations in the third position of triplets are highly profitable for protein synthesis. Since the genetic code is largely redundant, there is still space for some hidden secondary functions of this code. Numerical tests were obtained together with the Institute of Cellular Biology and Pathology of Charles University, and the results were published in [376822].

In [435691] we established a nonlinear dependence of the sudden colour changes on variable illumination and observation angles on wings of Apatura butterflies. Using Bragg's law, we performed several hundred angle measurements of a four-parameter problem to find this nonlinearity. This result has been obtained together with the Faculty of Science and the Institute of Cellular Biology and Pathology of Charles University, where all microscopic pictures were acquired.

We also proposed a new method for the reduction of a deterministic model of the biochemical system of circadian clocks. The reduction is based on a time delay quasi steady state assumption and we show its superiority over the traditional approach. This result was published in proceedings of an international conference.

Further, we have been interested in the number of possible stationary solutions of a system of two reaction-diffusion partial differential equations undergoing Turing instability. We used highly efficient Fourier spectral method combined with a fourth-order Runge-Kutta method to compute many samples of stationary solution and we have found that apart from the problem symmetries the number of possible stationary solutions is relatively low. This result was published in proceedings of an international conference.

It is worth mentioning that applications of mathematics in biology are a highly collaborative interdisciplinary research. Our team intensified this research since 2013 and several results we worked on have been submitted in 2014, but have not been published yet. For example, we prepared a general paper about the method of delay quasi steady state assumptions as well as a general paper about the unilateral regulation in systems undergoing Turing instability. Similarly, we submitted a paper about the multimodality of stochastic chemical systems and about the limitations of the Fokker-Planck equation to capture it. Further, we submitted a paper about the tensor parametric analysis of chemical systems based on the solution of higher dimensional Fokker-Planck equation. In some special cases, we succeeded to solve this partial differential equation even in 20 dimensions.

Inspired by the long-term collaboration with Lawrence Somer (Catholic University of America, Washington, D.C.), we investigated the structure of directed graphs that are associated with the congruence $x^k \equiv y \pmod{n}$. We succeeded to publish the results of this research in [360532].

Research Report of the team in the period 2010–2014

Institute	Institute of Mathematics of the CAS, v. v. i.

Scientific team	Evolution Differential Equations

1. General overview

The activities of the team EDE in the period 2010-2014 were focused on mathematical modeling of evolution processes in continuum mechanics and thermodynamics. The crucial issue in this discipline is to analyze the possibility of reliable predictions based on the physical balance equations subject to an incomplete set of data. This can be done under suitable structural assumptions on the empirical constitutive relations, and different modeling situations require using different techniques. The mathematical approach starts with the existence of solutions to given boundary value problems as a first test for the validity of the model, and passes to the study of qualitative properties of the solutions and their comparison with physical expectations, such as regularity, asymptotic behavior when some physical quantities converge to some singular values, long time behavior, or justification of simplified engineering models when “small" terms are neglected.

The core of the team is formed by experienced specialists in various domains: Compressible fluids (E. Feireisl, A. Novotný), incompressible fluids (J. Neustupa), fluid-structure interaction (Š. Nečasová), dynamics of mixtures and hydraulics (I. Straškraba), phase transitions (H. Petzeltová), material memory and hysteresis (P. Krejčí), unilateral problems (M. Kučera, J. Jarušek), function spaces (B. Opic, M. Krbec, A. Kufner), potential theory (D. Medková), but also young motivated scientists are part of the team.

Two team members received three prestigious national scientific awards. A. Kufner (Commemorative Medal of the University of West Bohemia, Plzeň, 2013, for his long-term work for the University and its development, and Honorary Medal of the Czech Mathematical Society, 2014, for his lifelong contribution to the development of the Czech Mathematics and of the Union of Czech Mathematicians and Physicists), and P. Krejčí (Bernard Bolzano Honorary Medal for Merit in the Mathematical Sciences in 2014).

2. Mathematical fluid dynamics

Main effort has been invested and main progress has been achieved in various directions of research in the mathematical theory of fluid flow. It has also received substantial support when E. Feireisl obtained his ERC Advanced grant in 2012. In the theory of compressible heat conducting fluid flow, the concept of weak and very weak solution, introduced previously by E. Feireisl and his collaborators, turned out to be a very fruitful source of inspiration for future research. It is based on the assumption that for nonsmooth processes, it makes no sense to try to control the local energy exchange between heat and macroscopic mechanics, and only a global energy balance is imposed in agreement with the First principle of Thermodynamics. Instead, the Second principle of Thermodynamics postulating nonnegative entropy production has to be fulfilled locally. Although the value of the entropy production cannot be prescribed for each process and the Second principle is supposed to hold only as an inequality, this information allows for deriving a whole bunch of results that had been inaccessible before. The idea has further been developed in [382804, 376740, 369769, 392313, 376973], where it was shown that the full Navier-Stokes-Fourier system possesses the property of weak-strong uniqueness, that is, if a regular solution to the system exists, then it necessarily coincides with the weak solution emanating from the same initial conditions. In other words, a possible regular solution is unique within the class of weak solutions. The proof is based on the concept of relative entropy inequality, which is in turn related to the so-called ballistic free energy and allows to control the distance between the weak and the strong solution. The case of fluids with density-dependent viscosity was treated in [393487].

Fluid properties in engineering practice are often expressed in terms of dimensionless numbers characterizing specific flow regimes (Mach number for the flow speed versus speed of sound, Reynolds number for turbulence, Rossby number for rotational flow, Péclet number for the heat conduction versus convection, Froude number for the competition between external and internal fields, etc.). A rigorous analysis of many practically important cases where some of these numbers tend to their limit values has been carried out by E. Feireisl and his collaborators in [380309, 434571, 333106, 340507, 393464]. Brinkman-type penalization is shown to converge in [353355]. This systematic study has brought a new insight into the topic and specified conditions for a rigorous justification of common engineering applications. Questions about long time behavior [349245] or time periodicity [376828] of solutions to the full Navier-Stokes-Fourier system have received appropriate answers which confirmed general physical opinion.

Š. Nečasová and her collaborators published a series of papers [369702, 376823, 393486, 342836, 430352, 430356] devoted to the investigation of the full fluid flow system in presence of radiation from different points of view (existence of global solutions in 3D, singular limits, asymptotic behavior in 1D). Here, the concept of compactness of velocity averages adapted to this particular situation has played a substantial role.

The problem of a physically proper choice of boundary conditions for fluid flow problems in Euler coordinates also received appropriate attention in the works by Š. Nečasová, E. Feireisl and others. It was shown in [349312, 398436] that complete slip boundary conditions on a domain with rough boundary generate a new type of boundary conditions on the regular limit boundary when the rugosity asymptotically vanishes. The case of moving boundaries was treated in [427471, 382809]. Other important questions related to finer qualitative properties of Navier-Stokes equations under various boundary conditions are studied by J. Neustupa et al. in [343551, 375190, 389760, 348177, 343538, 393111], in particular continuous dependence on the viscosity coefficient and an Euler type vanishing viscosity limit, and relationship between stability of the solution and spectral properties of the associated linear operator. New results have been obtained for partial regularity of solutions to the Navier-Stokes equations for incompressible fluids. Local regularity estimates in a small space-time neighborhood of a singularity have been derived in [376827, 431406]. Conditional regularity in the whole if some components of the solution are known to be regular is established in [428441, 380310]. Regularity of solutions to the stationary flow problem with a nonlinear constitutive law was established by V.

Mácha under different assumptions in [430423, 434078]. Optimal shape of the domain boundary under different flow regimes inside the domain and different cost functionals was the main subject of investigation carried out by J. Stebel in [373951, 436795, 356166, 378905, 430330].

Š. Nečasová initiated systematic studies about interactions between moving fluid and immersed solid bodies that gave rise to many publications with active participation of other team members (J. Neustupa, E. Feireisl, and A. Novotný) [430372, 395491, 359868, 357503, 364305, 373955, 364666, 359793, 399542, 345048, 435828, 342842, 376643]. They include fluid flow models past an obstacle or around a moving (in particular rotating) body, motion of a self-propelled body, motion of several rigid bodies under the influence of gravity, to name only a few. The problems range from basic questions of well-posedness of the model, over asymptotic properties of solutions and optimal decay estimates, to numerical studies of the underlying equations, which are mostly of Oseen or Stokes type. A theoretical background for these studies has been established by Š. Nečasová, D. Medková, J. Neustupa and collaborators in the papers [427503, 345654, 349614, 434080, 397809, 399394, 398062] devoted to various fine analytical properties of the Oseen and Stokes operators under different boundary conditions and in different function spaces, including a generalized maximum modulus principle for the Stokes and Oseen systems. Analytical and numerical studies of equations describing the flow of fluids with less standard constitutive relations are carried out by J. Stebel, O. Kreml et al. in [371221, 377496].

Other research topics are studied by H. Petzeltová and E. Feireisl and are related to propagating fronts and traveling waves in nonlinear convection-diffusion equations that have been constructed in [434174, 369968]. A liquid crystal model is proposed and its analytical properties including the long time behavior of solutions are analyzed in [349250, 374133]. The problem of compressible flow with a possibility of voids in 1D described by a degenerate system of PDEs was solved by I. Straškraba e.g. in [352455].

3. Mathematical modeling in solid dynamics

One research area within solid dynamics modeling is related to material memory and coordinated by P. Krejčí. A model for fatigue accumulation in elastoplastic has been proposed and studied e.g. in [394927, 434091]. The main idea is motivated by the engineering so-called “rainflow" method of cyclic fatigue evaluation which counts closed loops in the strain-stress diagram. In fact, this algorithm counts dissipated energy during each cycle, so that the dissipation rate can be taken as a measure for fatigue accumulation rate. The full system of momentum and energy balance equations in transversally vibrating beams and plates subject to material fatigue is shown to be well-posed here.

Hysteresis-based constitutive models for multifunctional materials with memory have been systematically studied during the evaluation period. A model for magnetostrictive hysteresis based on thermodynamic principles is proposed in [394926]. It manifests good agreement with all magnetoelastic experiments and establishes an explicit formula for the exchange between mechanical and electromagnetic energies, as well as for the hysteretic energy dissipation. Well-posedness of classical hysteretic models for shape memory alloys in various settings is discussed in [369988, 360070, 349601]. It is shown for example that a standard 1D dynamic problem for the original Souza-Auricchio model is ill-posed in the sense that a strong solution may not exist for some realistic initial data, and weak solutions exhibit negative entropy production and thus violate the Second principle of Thermodynamics. Real time control of piezoelectric sensors and actuators is the subject of papers [359278] and [374841], where an explicit inversion formula is derived for the piezoelectric constitutive operator accounting for both hysteresis and creep effects. It offers a powerful tool for engineering control algorithms.

Attention has been paid to mathematical modeling of solids in unilateral contact. J. Jarušek in his papers [399418, 359280, 391365, 397814] proves the existence of solutions of the contact problem in the cases of elastic or viscoelastic plates and shells. In [389746] he studies a model of contact of a body with an almost rigid obstacle allowing for some strictly limited interpenetration, which was completely missing in the literature; this is the first contribution in this direction. Interpenetration is to be interpreted as flattening of small surface asperities and/or filling possible microscopic holes. An infinite boundary traction is necessary in the model. An existence and uniqueness result is proved for the static problem with or without friction. Sharp pointwise necessary optimality conditions are derived in [356042] for optimal control problems governed by strongly monotone variational inequalities, whose solution map is directionally differentiable in the control variable, provided no control or state constraints are imposed. In presence of such constraints, optimality conditions in a “fuzzy” form are proved. Other types of contact problems are studied in [348169, 345046, 349600]. Shape optimization for elastic structures in contact was done by J. Stebel with collaborators in [430329]. Alternatively, P. Krejčí in [394925] described the contact of an elastoplastic body with a deformable elastoplastic obstacle by means of hysteresis operators in the bulk and on the boundary. The resulting system is shown to admit a unique strong solution, and the proof relies in a substantial way upon the Lipschitz continuity of the boundary contact hysteresis operator.

Modeling phase transitions has been another research topic under investigation here, mainly by H. Petzeltová and P. Krejčí. For nonlocal phase field systems with nonlocal interaction and a smooth phase potential with controlled growth on the boundary of the admissible domain, it is shown in [358300, 351228] that the solution remains separated from the pure phases and regular for all times. Furthermore, a nonsmooth generalization of the Lojasiewicz theorem established earlier in 2004 by E. Feireisl and H. Petzeltová is used here for proving that the whole solution trajectory converges to a single equilibrium. Well-posedness of a multi-phase transition problem with full dependence of the material parameters upon both the phase and the temperature, and with a van der Waals nonlocal interaction term and a general phase potential is proved in [360085]. Models for solid-liquid phase transitions with volume change (as, e.g., in water-ice mixtures) in a deformable container are derived and solved in [349653, 351138, 383891]. A nonstandard phase segregation model proposed by P. Podio-Guidugli is studied in [430395, 430402, 428614] from the point of view of convergence of time discrete approximations, continuous data dependence, and singular asymptotic limit.

4. Reaction-diffusion systems and unilateral problems

Reaction-diffusion systems describing spatial patterns in models in biology (e.g. embryogenesis) have been studied by M. Kučera and his collaborators. The influence of unilateral constraints describing sources which are active only if the concentration decreases below a prescribed value were investigated. Location of bifurcations of stationary spatially nonhomogeneous solutions (spatial patterns) was described in various situations [336125], [376831]. For instance, it was shown that in some cases, spatial patterns can occur for arbitrarily large ratio of diffusions of activator and inhibitor [374182]. This is a surprising and essential difference from standard situations (without unilateral sources) where a sufficiently small ratio of these diffusion coefficients is necessary for Turing instability and forming of spatial patterns.

Furthermore, qualitative properties of the Signorini problem were studied. A smooth dependence of solutions and contact regions on parameters was proved, existence of smooth local bifurcation branches was shown [337047], [377965], [354842] and their direction and stability were described. The results concern rather particular situations (one simple equation and two-dimensional case) but they are the first of this type for problems where the contact set changes with parameters.

5. Theory of functions and function spaces

The 1977 edition of the book “Function Spaces" by A. Kufner, O. John and S. Fučík (KJF) has become a classical reference monograph in applied mathematical analysis (almost 500 citations in MathSciNet). The monograph [397224] is the first part of a substantially extended version of KJF. It provides a complete up-to-date survey about spaces of integrable functions and is thus unique in the existing literature. Moreover, it serves as a necessary background for the investigation of Sobolev-type spaces. The second volume is in preparation. M. Krbec in a series of papers with coauthors proved deep theorems on optimal dimension-free embeddings for Sobolev spaces [375017, 375019, 422358, 430335], embeddings for weighted Sobolev spaces [374917, 358297], as well as trace theorems and Gagliardo-Nirenberg-type inequalities in Orlicz-Sobolev spaces [336111, 395505]. Another result [430331] is related to optimal regularity of solutions to the heat equation in Orlicz-Sobolev spaces with initial data from an Orlicz-Slobodetski space. A. Gogatishvili and B. Opic proved in [342833, 335002] a series of new embedding theorems for Bessel potential spaces, generalized Hölder spaces, Zygmund-type spaces, and Sobolev-Orlicz spaces. Boundedness and other properties of integral operators in weighted spaces were studied by B. Opic and J. Rákosník in [348195, 342853, 342832]. Hardy-type inequalities and their relationship with spectral theory for nonlinear operators were studied by A. Kufner in [429368, 394890, 343844, 383887, 387095].

6. International cooperation

Altogether 94 foreign experts of many nationalities participated in the publication activities of the team members. Among countries with the most frequent contacts we may cite France (22 different coauthors), Italy (15), Germany (15), USA (12), and Poland (6). The other collaborators came from Portugal, Ireland, Great Britain, China, Spain, Slovakia, Norway, Sweden, Korea, Chile, Venezuela, Tunisia, Jordan, and Kazakhstan.

Team members have intensive contacts with their colleagues in many countries and are often invited to foreign academic institutions. In the period 2010–2014, they delivered lectures at universities and research institutes in Germany (Berlin, Dresden, Bonn, Leipzig, Heidelberg, Darmstadt, Duisburg-Essen, Bielefeld, Göttingen, Munich, Münster, Mainz, Kassel, Hamburg, Jena), France (Paris, Marseille, Toulon, Lyon, Corte, Perpignan), Italy (Rome, Milan, Naples, Torino, L'Aquila, Benevento), Austria (Vienna, Klagenfurt, Innsbruck), Spain (Madrid, Bilbao), Poland (Warsaw, Poznań, Katowice), China (Beijing, Nanjing, Shanghai), Hungary (Budapest), Slovakia (Bratislava), Great Britain (Oxford), Japan (Tokyo, Kyoto), Korea (Seoul), India (Bangalore), Romania (Cluj-Napoca) and Georgia (Tbilisi). In addition, the EDE team members delivered over 130 plenary or invited lectures in international congresses or conferences.

With substantial active participation of the team members, 13 international workshops, conferences, and summer schools were organized in 2010-2014. Among them, the most important ones were the 6th European Congress of Mathematics in Krakow with almost 1000 participants, where E. Feireisl was the chairman of the Scientific Committee, and the international conference Equadiff 13 in Prague with over 370 participants, where E. Feireisl was the head of the Scientific Committee and P. Krejčí was the head of the Organizing Committee.

Research Report of the team in the period 2010–2014

Institute	Institute of Mathematics of the CAS, v. v. i.
Scientific team	Logic and Theoretical Computer Science

The research in the group is centred around complexity theory and related areas of mathematical logic and theoretical computer science. The study of the complexity of computationally hard problems is one of the leading motivations of contemporary theoretical mathematics and computer science. It comprises both theoretical foundations for methods that aim to demonstrate that certain problems are indeed hard, and algorithmic methods that provide efficient methods for at least partial or satisfactory solutions to problems that perhaps cannot be solved optimally. The group was active in both these areas of research.

On the foundational side, our group is one of the world's leading centres of research in bounded arithmetic and proof complexity. It also has strong researchers in classical complexity theory (circuit and communication complexity, pseudorandomness, and connected areas in combinatorics). The aim of these areas is to understand combinatorial and logical questions and the difficulty of their algorithmic solutions. This is motivated by basic questions such as the famous “P=NP?” problem. An overall goal is to provide evidence that some problems do not have a fast algorithmic solution. These questions have foundational importance, but potentially may also have practical applications in areas such as automated theorem proving (e.g., by showing limits of the used methods) and data security.

The foundations of mathematics are most often formulated and studied using set theory. Part of the team works in this area, working in particular on forcing, the complexity of equivalence relations, and independence results. There are many themes in common with bounded arithmetic, and we report some research below drawing on ideas from the two fields.

On the algorithmic side, the research focused on online algorithms (which provide good solutions for problems whose input arrives over time), databases, controlled formal models and supervisory control of discrete-event systems.

We present below a selection of the main results obtained during the evaluation period, divided into sections on logic, set theory, theoretical computer science, control theory and combinatorics.

1. Logic

Researchers working in logic (excluding set theory) were Emil Jeřábek, Jan Krajíček (for part of the evaluated period), Pavel Pudlák, Neil Thapen, and for part of the period a postdoc Massimo Lauria. Recently Pavel Hrubeš joined the team after a stay in the USA and Canada (including a year at the IAS in Princeton).

The main topic of research was proof complexity. This topic consists of two closely connected areas. The first one, *propositional proof complexity,* studies proofs as a particular kind of finite combinatorial object and looks for lower bounds on their size (or on other complexity measures), and is closely connected with the history of computational complexity: a very early version of the P vs NP problem is a question posed by Gödel about searching for proofs. The central problem in this area is the original formulation of NP vs coNP, due to Cook: is there a propositional proof system in which every tautology has a polynomial size proof? The second current of research goes back to foundational questions about the strength of weak fragments of Peano arithmetic, in particular the theories known as *bounded arithmetic.* These theories aim at capturing the informal concept of "feasible" reasoning in a way analogous to how the polynomial time hierarchy captures the notion of a feasible algorithm. It is possible to translate between first-order proofs in these theories and short proofs in commonly studied propositional proof systems, and there are close connections with algorithmic complexity: we can measure the "algorithmic strength" of a theory by studying the complexity of the functions that it proves are total.

During the evaluation period, Pudlák was awarded the ERC grant FEALORA to work on proof complexity (with Hrubeš, Jeřábek and Thapen as members of the team). He also published a monograph on the logical foundations of mathematics from the perspective of computational complexity [Springer 2013 393022]. This book has mostly the character of a survey, but also contains some of his own results and some new ideas about *feasible incompleteness*.

1.1. Propositional proof complexity

Recently, most of the research in this area has focused on refining information about weak proof systems for which we do have lower-bound methods. In contrast, Krajíček worked on developing methods that could prove lower bounds for stronger proof systems (and thus also separations of the corresponding first-order theories), and has developed an original approach based on constructing models of arithmetic using forcing techniques and recent results in computational complexity. This has appeared in a monograph [Cambridge Univ. Press 2011 369674].

The best-understood weak system is *resolution*,[1] which is of independent interest for automated theorem proving. The most studied measures of the complexity of a propositional proof are its length, its width (the maximum number of variables in any formula) and its space (the smallest size of blackboard we can present it on, if we are allowed to erase formulas when we no longer need them). In this direction our group has achieved several interesting results. Bonacina, Galesi, and Thapen [FOCS 2014 438303] proved a lower bound on total space (the total number of symbols that must fit on the blackboard) for resolution, showing that a random *k*-CNF requires quadratic space. Filmus, Lauria, Nordström, Ron-Zewi, and Thapen [CCC 2012 385827], showed space lower bounds for formulas with small initial width in the polynomial calculus proof system. Galesi, Pudlák and Thapen [to appear at CCC 2015], for the cutting planes proof system, showed that every tautology can be proved in space five (measured by the number of formulas that must fit on the blackboard), and also proved space lower bounds for a restricted system. Thapen [submitted 2014] showed that there is a tautology that has short proofs in resolution, and also has proofs with small width, but for which any proof with small width must have exponential length. Space is a measure of particular interest because of possible connections with the memory requirements for automated theorem proving, and the results about space above resolve several open questions from the seminal 2002 paper on space by Alekhnovich, Ben-Sasson, Razborov and Wigderson.

Formalizations in weak systems sometimes enable one to prove unexpected results. A nice example of this kind of result is a work of Beckmann, Pudlák and Thapen [ToCL 2014 430389], who used a formalization in bounded arithmetic to show that if there is a

[1] More precisely, propositional resolution.

polynomial time algorithm which detects whether a tautology has a short resolution refutation, then parity games can be decided in polynomial time. Whether either of the two problems possesses a polynomial time algorithm are well-known longstanding open problems.

Another important problem is to find candidate hard-to-prove tautologies. In this context tautologies based on the finite Ramsey theorem have been extensively studied. Lauria, Pudlák, Rödl, and Thapen [ICALP 2013, to appear in Combinatorica], came up with a new family of tautologies connected with the Ramsey theorem and studied the length of proofs of them in resolution. They proved that for every graph that does not have cliques or independent sets of size bigger than 2log *n*, a tautology expressing this property only has resolution proofs of superpolynomial size. It is a famous open problem, posed by Paul Erdős long ago, to find explicit constructions of such graphs, and this result may also be viewed as an explanation of why such constructions are difficult.

1.2. Bounded arithmetic

The main problem in bounded arithmetic is to prove separations, that is, that theories that use stronger induction axioms are indeed stronger. These problems seem to be even more difficult than the corresponding problems about separations of complexity classes. However, one can prove separations relative to an oracle, although in general this is also more difficult than in computational complexity. Furthermore, we are particularly interested in separations by low complexity sentences, in particular by the *provably total NP search problems* of a theory, which provide a good measure of the mathematical tasks that a theory can carry out.

To this end, it is necessary first to characterize the search problems of these theories. It took a long time to find such characterizations for all levels T_2^k of the bounded arithmetic hierarchy. The first such characterization was given by Thapen together with our former postdoc A. Skelley [Proc. London Math. Soc. 2011 369680]. Another characterization was given by Pudlák and Thapen [APAL 2012 374023], who showed that T_2^k induction is equivalent to an axiom asserting the existence of alternating sequences of maxima and minima of length *k*, and using this showed that the search problems of T_2^k are exactly those reducible to the task of finding a Nash equilibrium in a certain *k*-turn game. Buss, Kołodziejczyk, and Thapen [JSL 2014 433869], and with Atserias [ToCL 2014 437494], came up with a new approach to separation questions by looking at the search problems of the theory of approximate counting developed by Jeřábek in 2004-2009, which is similar to T_2^2 in strength but is not based directly on induction, and which captures many common counting arguments from finite combinatorics. They proved limits on the strength of its natural subtheories, relative to an oracle. This gives some new understanding of the simplest unknown question about separation, between T_2^3 and T_2^2.

Jeřábek's work in the evaluation period illustrates the connections between propositional proof complexity, bounded arithmetic, circuit complexity and combinatorics. The monotone sequent calculus proof system MLK is known to quasipolynomially simulate the usual sequent calculus system LK, by work of Atserias, Galesi and Pudlák, but it is open whether it polynomially simulates LK. Jeřábek made progress on this problem by introducing a theory of bounded arithmetic corresponding to a variant of the complexity class NC^1, and proving that it can formalize the construction of the Ajtai–Komlós–Szemerédi sorting network [APAL 2011 353276]. This reduces the polynomial simulation of LK by MLK to formalization of the existence of suitable expander graph families in an NC^1 theory.

The elementary integer operations +, · and < are computable in uniform TC^0. Jeřábek studied the question of which interesting properties of these operations are provable in the corresponding arithmetical theory VTC^0. He showed that a certain strengthening of VTC^0 proves induction for quantifier-free formulas in the language {+, ·, <} [Arch. Math. Log. 2015]. As part of this work, he obtained the purely computational result that approximations of complex roots of constant-degree polynomials are computable in uniform [Th. Comp. Sci. 2012 382374].

As another connection of bounded arithmetic to computational complexity, Jeřábek exploited his earlier results on provability of the quadratic reciprocity theorem in a certain arithmetical theory to show that integer factoring (as a function problem) is in the complexity class PPA assuming a generalized Riemann hypothesis, and that unconditionally it has a randomized reduction to PPA. Also, computation of square roots modulo arbitrary integers is in PPA (unconditionally) [submitted 2012].

Jeřábek also investigated recursive saturation of real closures of models of weak arithmetic in a joint work with Kołodziejczyk [Arch. Math. Log. 2013 388604]; encoding of sequences in the induction-free theory of discretely ordered rings [MLQ 2012 377734]; parameter-free induction schemata and induction rules in bounded arithmetic; and interpretability in existential theories.

1.3. Nonclassical logics

A basic problem encountered in automated theorem proving, knowledge representation, and other areas of computer science, is equational unification: given terms *t* and *u*, determine whether there exists a variable substitution *s* (a unifier) that makes *s*(*t*)=*s*(*u*) valid in a background equational theory (variety), and if so, describe a complete set of such unifiers. Unification is, inter alia, studied for varieties arising from the semantics of nonclassical propositional logics (e.g., modal logics), where it is closely connected to the admissibility problem: can we extend the consequence relation of the logic by a given inference rule *A*/*B* without changing the set of derivable formulas? In terms of universal algebra, admissible rules describe the universal theory of free algebras in the corresponding variety. A major problem in this area is decidability of unification and admissibility in the basic modal logic **K**. Jeřábek proved an important negative result concerning this problem: unification in **K** is badly behaved, namely it has the worst possible (nullary) unification type [J. Log. and Comp., to appear].

Admissibility of rules is well understood for transitive modal and superintuitionistic logics with certain frame-extension properties, but only scarce information is available on other nonclassical logics. In order to remedy this situation, Jeřábek investigated admissible rules of Łukasiewicz logic, which is one of the most important multivalued logics – its semantics is quite different from modal logics, and it has a nontrivial admissibility problem. In particular, he proved that admissibility in Łukasiewicz logic is decidable and PSPACE-complete, gave a characterization of admissible rules in terms of geometry of the underlying polyhedra, and constructed bases of admissible rules [J. Log. and Comp. 2010, 2010, 2013, 343478, 351058, 392459].

In unification theory, it is customary to consider unification with free constants (parameters), however, substantial parts of the theory of admissibility in transitive modal logics only deal with parameter-free rules and unifiers. Jeřábek, working on a series of papers, generalized these results to rules with parameters, including semantic description, complete sets of unifiers, bases of rules, and computational complexity of admissibility [APAL, to appear].

2. Set theory

Prague has traditionally been a center of set theory, with focus on forcing and descriptive set theory. There has been an intense collaboration with the set theory group at Charles University in Prague and the nearby Kurt Gödel Research Center for Mathematical Logic at the University of Vienna. Further, there has been interactions with members of the Functional analysis group of the Institute (Bohuslav Balcar, in particular). The researchers in the group were David Chodounský and Jindřich Zapletal (who held the prestigious Purkyně Fellowship), and a PhD student Michal Doucha.

2.1. Forcing

Famously introduced in 1963 by Paul Cohen as a tool of proving the independence of the Axiom of Choice, *forcing* has been a central tool used to advance our understanding of the foundations of mathematics (in particular, for proving consistency and independence results) since. Forcing is commonly used to answer questions stemming from point-set topology, and much of the work of the group focused in this direction. Let us now describe highlight several particular results of the group in the evaluation period.

The motivation driving the research of Chodounský is the effort to understand the combinatorial structure of the boolean algebra $\Pi(\omega)/Fin$, which is one of the main direction of research in the contemporary set theory of the reals.

In 1978 Balcar and Frankiewicz showed that for each pair of distinct infinite cardinals kappa and lambda, the quotient Boolean algebras $\Pi(\kappa)/Fin$ and $\Pi(\lambda)/Fin$ are not isomorphic. The only possible exception in their work was the case when $\kappa = \omega$ and $\lambda = \omega_1$. Now, almost 40 years of intensive work on the problem (now called the after the town of Katowice), we still do not know whether it is consistent that $\Pi(\omega)/Fin$ and $\Pi(\omega_1)/Fin$ are isomorphic. The problem is widely regarded as one of the most interesting questions relating to the Čech-Stone compactification of omega. Chodounský [Topology Appl. 2012 380714] investigated consequences of the existence of such isomorphism, and provided a partial solution to the problem by demonstration the consistence of all known consequences holding in one model simultaneously. Namely, he used the method of countable support iteration of proper forcings to build a model of the Zermelo–Fraenkel set theory with a strong-Q-sequence (an uniformizable almost disjoint system) of size together with a dominating family of functions of the same size. This is currently the best result about the Katowice problem.

Continuing the research on combinatorics of $\Pi(\omega)/Fin$, Borodulin-Nadzieja and Chodounský [Fundamenta Mathematica, 2015] investigated the structure of towers (chains of subsets of the integers well ordered by almost inclusion). The characterization of various types of towers turned out to be useful for investigation of gaps on integers. They constructed several examples of gaps, including a forcing indestructible gap not satisfying the Hausdorff condition, thus answering a classical question from the theory of gaps (Scheepers [Set theory of the reals, 1991]).

To solve questions arising in the previous work on towers, Chodounský worked on a problem typical in the theory of forcing; diagonalizing filters in $\Pi(\omega)/Fin$ with forcing notions with controlled bounding-like properties. His work with Repovš, and Zdomskyy [J Symb Logic, to appear] connected topological covering properties of filters with bounding like properties of typical forcing notions used for their diagonalization. These results provided tools which turned out to be useful even in a more general setting, solving several open problems about forcings connected with filters on $\Pi(\omega)/Fin$.

Zapletal resolved a 25 year old question of David Fremlin concerning forcing, by applying a construction from infinite dimensional topology in a novel way. He described how properties of sigma-ideals of compact sets from mathematical analysis correlate with the properties of forcing notions derived from the ideals.

Chodounský and Zapletal [Annals Pure App Logic, to appear] defined a new type of forcing properties derived from properties of subsets of complete Boolean algebras. The most important instance of this seems to be the property of being a centered set, and the derived notions of Y-c.c. and Y-proper forcings. They also developed a general theory of PSI-c.c. a PSI-proper forcings. This presents a new and fruitful approach to classification of forcing notions, and is likely to spawn interesting applications in future research.

2.2. Classification of analytic equivalence relations

The field compares the complexity of classification problems found all over mathematics. One recent advance refuted the longstanding classification program for C*-algebras (see e.g. [Farah: Logic and operator algebras. Proceedings of the ICM 2014]) by showing that the classification problem is the most complex possible in its very broad category. The immediate challenges include for example the evaluation of complexity of the Langlands classification program for unitary representations of countable groups. Zapletal's contribution to the field is divided into the following areas:

A. In a recent book by Kanovei, Sabok, and Zapletal, "Canonical Ramsey Theory on Polish Spaces'' [Cambridge Tracts in Mathematics 2013 422163] the authors analyze how analytic equivalence relations simplify when restricted to a large Borel set. The methods include associating sophisticated models of set theory to equivalence problems. One of many results therein states that for every equivalence relation on Hilbert cube reducible to isomorphism of countable structures there is a product of perfect sets on which the equivalence relation is reducible to identity.

B. In 2000, Greg Hjorth introduced the concept of turbulence as a means to classify Borel equivalence relations. This work won him and Alexander Kechris the 2003 Karp Prize of the Association for Symbolic Logic. Zapletal identified a forcing restatement of Hjorth's turbulence concept with new many generalizations. They make it possible to prove ergodicity results to the effect that any attempt at reducing a given equivalence problem to another one will necessarily have to be trivial (and therefore unsuccessful) on a large set of cases.

C. Zapletal also showed that many analytic equivalence relations have canonical extensions to the transfinite, where sophisticated tools of modern set theory can be used to discern the complexity of the problems. For example, there are equivalence relations such that the natural proof of irreducibility of one to another uses the consistency of the failure of the Singular Cardinal Hypothesis.

Details of results in B. and C. are contained in an expository paper [Analytic equivalence relations and the forcing method, The Bulletin of Symbolic Logic, 2013 430392] and upcoming book "Reducibility invariants in higher set theory''.

3. Theoretical computer science

Researchers in our group working in theoretical computer science were Dmitry Gavinsky, Pavel Hrubeš, Michal Koucký, Tomáš Masopust, Pavel Pudlák, Jiří Sgall (for a part of the period) and PhD student Jan Bulánek. Our main topic in theoretical computer science has been computational complexity, specifically, circuit complexity, communication complexity, pseudorandom generators and algebraic complexity. Furthermore, researchers in the group studied the complexity of various algorithmic problems.

3.1. Circuit complexity and pseudorandom generators

Gal, Hansen, Koucký, Pudlák, and Viola studied bounded depth circuits computing good error correcting codes and pair-wise independent hash functions [STOC 2012 386309, IEEE Trans. Inform. Th. 2013 422134]. They found optimal bounds for such circuits. The main tool in their work was analysis of properties of a certain kind of well connected graphs that are called super-concentrators. Such graphs were analyzed before. It turns our that the previous analysis implicitly studied two distinct classes of super-concentrators. This work identified an intermediate class between the two and established their relationship. Circuits computing good error-correcting codes correspond to this intermediate class of super-concentrators and have slightly different size requirements than the other two super-concentrator classes.

Paturi and Pudlák, studied the complexity of Boolean circuit satisfiability [STOC 2010 351120]. It has been conjectured that this problem is exponentially hard in the strong sense, but no results supporting this hypothesis were known. In this paper we present a technique to justify this hypothesis. For algorithms that run in polynomial time and solve the given problem with positive probability, we show that if circuit satisfiability can be solved with slightly more than exponential probability, then it can be solved much more efficiently, which is unlikely. Koucký, Nimbhorkar, and Pudlák constructed pseudorandom generators of polynomial size for permutation branching programs and group products [STOC 2011 371063]. Construction of pseudorandom generators is a major open problem and the current research effort focuses on tackling restricted classes of computational models and algorithms. Our result was a substantial progress in the research area.

Separating P from NC^1 (the class of all functions that can be computed by log-depth and poly-size Boolean circuits) is one of the most fundamental open questions in computational complexity. Gavinsky, Meir, Weinstein and Wigderson, developed a new approach based on information complexity and used it to make an important step towards resolving this question [STOC 2014 434460].

In the short period in the institute Hrubeš was finishing several projects which started as a postdoc in North America. Namely, a work on Boolean circuits with gates of unbounded fan-in with Rao [to appear at CCC 2015], a work on arithmetic proof complexity with Tzameret, a former postdoc in our institute [accepted in SIAM J. on Comp.], and a work on non-commutative rational functions with Wigderson [submitted].

3.2. Communication complexity

Koucký with several co-authors studied questions in the field of communication complexity [CCC 2012 386317, CCC 2013 422288]. They obtained results in the area of privacy and in the area of information complexity. These are two current active research directions people work on in this field. In the information complexity setup they studied the relationship between using private and public randomness and they established certain conversion results. In the area of privacy they established trade-offs between the amount of communication and the privacy loss as measured by so called privacy approximation ratio. They also established connection of this measure to other previously studied notions of privacy and to information complexity.

Bavarian, Gavinsky, and Ito, studied the power of different sources of shared randomness in the setting of communication complexity. Shared randomness is access to correlated random bits, and it is one of the basic and most important communication resources. They introduced a natural measure for the strength of the correlation provided by a bipartite distribution, argued its usefulness and shown that shared randomness forms a proper hierarchy with respect to this measure [ICALP 2014 434098].

Gavinsky, Ito, and Wang, studied the role of shared randomness in the context of multi-party communication protocols – the regime where more than two parties are involved in solving a communication task. They demonstrated a hierarchy (with exponential separations between the levels) of modes of shared randomness, with the usual shared randomness (where all parties access the same random string) as the strongest form in the hierarchy. Even more surprisingly, they showed that the resource of quantum communication does not, in general, allow to solve efficiently even a functional problem that can be solved by a three-bit three-party classical protocol that uses shared randomness [CCC 2013].

Gavinsky and Lovett, investigated the log-rank conjecture, proposed by Lovász and Saks in 1988. The conjecture suggested that for any function, its deterministic communication complexity is polynomially related to the logarithm of the real rank of the associated matrix. They proved that several natural (and widely studied) measures in communication complexity are equivalent, up to polynomial factors in the logarithm of the rank of the associated matrix: deterministic communication complexity, randomized communication complexity, information cost and zero-communication cost. Therefore, in order to prove the log-rank conjecture, it is sufficient to show that low-rank matrices have efficient protocols in any of the aforementioned measures [ICALP 2014 434135].

3.3. Online algorithms

Bulánek and Koucký, worked on the *online labeling problem*. In a sequence of papers, together with Saks and two other students, Babka and Čunát, they essentially completely resolved its complexity [STOC 2012 386313, ESA 2012 382019, ICALP 2013 395301]. The online labeling problem is a fundamental algorithmic problem in which one tries to maintain an array of sorted elements. New items are arriving that need to be inserted in the array. That may necessitate to move elements that are already stored in the array. The question is what is the best strategy that minimizes the average number of moves when $O((\log n)^2)$ we do not know the sequence of the items in advance. Algorithmic solutions that achieve moves per item on average are known since early 80's and it has been open since then whether they can be improved. Bulánek, Koucký and their coauthors show that the known algorithms cannot be improved as the cost of order $(\log n)^2$ is necessary. The actual complexity differs depending on the relationship between the number of items and the array size and they establish optimal bounds for essentially all possible settings. This lower bound has consequences for design of good data structures and algorithms as well as in other settings.

Sgall worked on the classical problem of Best-Fit Bin Packing problem. In a series of papers he gradually improved bounds and eventually he, together with Gy. Dosa, found the optimal ratio, thus solving a more than 40 years old problem [STACS 2013 424750, ICALP 2014 440923]. (His last papers, however, were published with his new affiliation at Charles University.)

3.4. Databases and controlled formal models

Masopust with coauthors investigated the options to introduce an interface for the XML Schema, which is currently the only schema specification language widely accepted by industry [ICALP 2013 394007, MFCS 2014 431318]. The research is still in progress and the aim of the investigation is to transform XML schema specification languages from machine-readable descriptions to descriptions readable and understandable by humans. They investigated special kinds of regular expressions closely related to expression such as XPath, whose languages correspond to languages definable by fragments of first-order logic and are simpler for humans to understand than standard regular expressions. They showed that the problem boils down to the separation problem of (regular) languages by piecewise testable languages. They proved that the separation problem is decidable in polynomial time if the input is given as two nondeterministic finite automata and investigated some upper bounds on the complexity of the computation of a separator language. The results have also applications in logic, where they provide a new insight into the lowest levels of Straubing-Therien and dot-depth hierarchies, and can also find applications in query answering.

Many formal models existing in the literature are mostly not directly applicable in new areas. However, sometimes it is sufficient to add some additional restrictions on their behavior to increase (or reduce) their power. In the context of formal grammars, such restrictions usually involve restrictions on the derivation process, namely, the freedom the rules can be applied. Examples of such models based on context-free grammars are matrix grammars, random context grammars or program grammars. Dassow and Masopust studied context-free grammars with a simpler restriction on the application of their rules, and proved that these grammars still have the same generative power as most of the other variants studied in the literature (including those mentioned above). As a result, they improved known normal forms of several of those mechanisms [J. Comput. and System Sci. 2012 367493].

4. Control theory

Jan Komenda and Tomáš Masopust worked in the area of supervisory control of discrete-event systems. Supervisory control theory is a formal approach for control of logical automata with partial transition functions that aims to impose safety specification (e.g. avoidance of forbidden states) and nonblockingness (avoidance of deadlocks and livelocks). The ultimate goal of supervisory control is to develop formal theory that enables an automated synthesis of controllers that meet these goals by construction so that further verification is not needed. The main obstacle is the computational complexity, because the number of states of concurrent automata grows exponentially in the number of components, which prohibits global (centralized) computation of controllers. Moreover, there are decidability issues in control of timed systems and logical systems with partial observations.

The problem is that computationally cheap local controllers fail in achieving jointly a safety specification. In order to face this issue, we have proposed coordination control approach for large concurrent discrete-event systems, where global system is modeled by synchronous products of local automata. This approach may be seen as a trade-off between a purely local (modular) control synthesis that typically do not lead to sufficiently permissive supervisors and leads to blocking, and a global control synthesis that is too expensive from the computational complexity viewpoint. During the last five years coordination control approach has been developed and gradually generalized in several directions.

We have first derived necessary and sufficient conditions, called conditional controllability and conditional closedness, for the specification language to be achieved by the proposed coordination control architecture [Automatica 2012 373656]. In the case, where specification language fails to be conditionally controllable we have presented a procedure for distributed computation of the supremal conditionally controllable sublanguage of a prefix-closed specification [Automatica 2012 378943]. These existential and constructive results are fundamental and have been extended in several directions: to partial observations, to non prefix-closed specification languages, and several variants for the distributed computation of maximally permissive solutions have been built upon this initial result.

Very important is the extension to the multi-level coordination control based on hierarchical structure of groups of subsystems, their respective coordinators and supervisors. This is useful for complex systems composed of a large number of subsystems that are organized into groups on the lowest level of the hierarchy and with low-level coordinators assigned to each group. The high-level coordinator then ensures communication among these groups.

We have proposed two approaches in this multi-level framework. Top-down approach, which is based on the top-down computation of coordinators and supervisors are computed at the lowest level. Both existential results based on the multi-level counterpart of conditional controllability and constructive results enabling distributed computation of supervisors have been derived. In the bottom-up approach supervisors are computed in a bottom-up manner starting from the lowest level and computation of coordinators for nonblockingness is also carried out from the bottom level to the top level. We have shown that constructive results of multi-level coordination control can be applied to decentralized supervisory control with communicating supervisors, where the concurrent structure of the system is not given, but there are still controllers that observe and control different subsets of events.

In the area of timed discrete-event system (TDES) we have worked on supervisory control of (max,+)-automata and related classes of timed Petri nets. The main problem here is that an equivalent condition for having finite-state (max,+)-automata as controllers is unambiguity of (max,+)-automata, which is only slightly larger class of automata than deterministic (max,+)-automata. Therefore, we have concentrated most of our research effort on the problem of determinization of (max,+)-automata and related classes of timed Petri nets. We have proposed a structural condition and a logical (language based) sufficient condition for determinization of behaviours of timed Petri nets, called strong liveness that is very closed to the concept of bounded fairness widely studied in the literature. We have shown how bounded fairness can be imposed by supervisory control [J. Discrete Event Dynamic Systems 2015], which is important for TDES that do not admit a representation by deterministic finite state (max,+)-automata, because they can be determinized using supervisory control. This opens the way to modular and coordination control of timed Petri nets based on approximate determinization of their behaviours.

5. Combinatorics

The group aims at fostering the connections between theoretical computer science and combinatorics. Many problems stemming from theoretical computer science are combinatorial in nature. For example, Gavinsky and Pudlák defined a new type of an expander-like combinatorial structure, called *partition expanders*, and showed their applications in communication complexity [STACS 2014 434518]. Activity in this field increased after Jan Hladký joined the group in September 2014. During the four months within the evaluation period Hladký focused on pushing the quasirandomness-based methods he developed in [arXiv:1404.0697, Israel J. Math., to appear] towards the full solution of the Tree Packing Conjecture of Gyarfás from 1976.

Research Report of the team in the period 2010–2014

Institute	Institute of Mathematics of the CAS, v. v. i.

Scientific team	Topology and Functional Analysis

The research activity of the team was concentrated around functional analysis and related fields like general topology, complex analysis and applied mathematics. The spectrum of research interests of the team is large, as its members backgrounds are in analysis (Ambrozie, Engliš, Fabian, Gogatishvili, Hájek, Zizler), combinatorics (Müller), geometry (Kopecká), set theory (Kubiś), mathematical physics (Šilhavý), and matrix theory (Fiedler). There are two new young researchers (postdoctoral fellows) appointed in 2014 in the TFA department: M. Doležal and M. Kraus, who are actively participating in the scientific projects of the team.

The main lines of research were as follows:

1. Operator theory (Ambrozie, Kopecká, Müller)
2. Banach space theory (Fabian, Hájek, Kraus, Kubiś, Zizler)
3. Universal homogeneous structures (Doležal, Kubiś)
4. Function spaces (Gogatishvili)
5. Complex analysis (Engliš)
6. Thermomechanics of deformable solids (Šilhavý)

1. Operator theory

The research in operator theory was concentrated mainly to the study of orbits of operators. By an *orbit* of a bounded linear operator $T: X \to X$ acting on a Banach space X we mean a sequence $\{T^n x: n = 0,1,...\}$, where $x \in X$ is a fixed vector. Similarly, a weak orbit is a sequence $\{\langle T^n x, x^* \rangle: n = 0,1,..\}$, where $x \in X$ is a fixed vector and x^* is a fixed linear functional. Analogously, orbits and weak orbits can be defined in other contexts: for mappings in Banach lattices, ordered vector spaces, operator semigroups, orbits of general sequences of operators, etc.

The notion of an orbit is very simple and natural. Orbits appear frequently in operator theory and in many other situations, e.g., in harmonic analysis, ergodic theory, differential equations, theory of dynamical systems, and in the theory of operator semigroups. For example: Fourier coefficients of a function with respect to a measure μ on the unit circle may be interpreted as a weak orbit of the corresponding unitary operator on $L^2(\mu)$; any orbit of a strongly continuous semigroup of operators provides a solution of the corresponding linear differential equation. Many important problems can also be formulated in the language of orbits and weak orbits. The best example of this kind is the famous invariant subspace/subset problem, i.e., the question whether each bounded linear operator on a Hilbert space has a non-trivial closed invariant subspace/subset. Indeed, an operator has no nontrivial closed invariant subset if and only if each nonzero vector has a dense orbit.

Research within this topic was leaded by V. Müller, an internationally recognized expert in the area of operator theory, and was partially supported by the FP7 Marie Curie Action "International Research Staff Exchange Scheme" project AOS: Asymptotics of Operator Semigroups (in this project V. Müller is a Scientist in Charge for the Institute of Mathematics CAS, coordinating institution is IM PAN in Warsaw).

Among the results obtained in this direction by V. Müller let us mention the properties of asymptotic behaviour of strongly continuous semigroups of operators. The techniques known in the discrete case do not work in the continuous setting. However, in [422128] it was shown that analogues of many results remain true for orbits of strongly continuous semigroups. The results have straightforward applications to the study of Fourier transforms.

Linear dynamics is a rapidly developing area of operator theory with close relations to theory of dynamical systems. Linear dynamics studies especially orbits that behave in a very irregular, chaotic way. The basic concept is the notion of hypercyclic vectors (vectors with dense orbits) and its variants (supercyclic vectors, mixing, weakly mixing, frequently hypercyclic vectors etc.). The notions of Li-Yorke chaos and distributional chaos were systematically studied by V. Müller and coauthors [395496]. These notions play an important role in the theory of dynamical systems and the paper introduced them to the context of linear dynamics.

Further research was concentrated on the study of joint numerical radii of n-tuples of operators. Questions in this direction can be formulated in the language of weak orbits. Although these questions are very natural, they have never been studied before. The research in this direction was originated by V. Müller in [425905] and continued in [431956].

Additionally, the following problem concerning convergence of iterating projections in the Hilbert space was considered. Namely, when sending a point back and forth by projections onto two subspaces of a Hilbert space, the iterates converge by a classical result of von Neumann. For a long time it was not known whether the same is true for projections onto at least three subspaces; this is the case in many particular situations. E. Kopecká and V. Müller have been successful in constructing an example of three subspaces of the Hilbert space together with a divergent sequence of iterations, thus giving the final answer to this problem and improving a similar example of five subspaces found previously by A. Paszkiewicz [434096].

Another topic studied in operator theory were weakly wandering vectors. This notion originated in ergodic theory. A classical result of Krengel known for unitary operators was generalized by V. Müller with Yu. Tomilov to a much more general setting of power bounded operators [360298].

Further research concerned the reflexive closures of spaces of operators. It is well known that any pair of complex matrices admits a nontrivial linear combination which is singular. This is equivalent to the non-emptiness of the spectrum, which in turns is equivalent to the fundamental theorem of algebra that each complex polynomial has a root. In [338965] this fact is generalized to an arbitrary number of matrices by describing the codimension of the intersection of the ranges of all their nontrivial linear combinations. The result is then applied to the reflexivity deficit of subspaces of operators; however, it may be useful in other situations as well.

2. Banach space theory

The research in this area was performed by the team members M. Fabian, an expert in topology and nonseprable Banach space theory, renormings and also in nonsmooth analysis and optimization; P. Hájek, V.Zizler, experts in nonlinear functional analysis, renormings and smoothness in Banach spaces, and W. Kubiś, working in applications of logic and category theory in functional analysis and topology.

2.1. Published compact sources of recent developments and classical results in Banach space theory

It is remarkable that in the Banach space theory, within the evaluation period, there were three monographs coauthored by the team members:

The foundations for the abstract theory of C^k-smoothness in infinite-dimensional real Banach spaces, and investigations of its intimate connections with the structural properties of the underlying spaces were exposed in the monograph coauthored by a team member P. Hájek [436754]. The main objects of the theory are polynomials and C^k-smooth (including real analytic) mappings. A large part of the theory was developed by P. Hájek. In particular, let us mention the following noticeable results:

- construction of bidual extensions of uniformly smooth mappings and their applications to the theory of weakly compact uniformly smooth mappings [389770]
- showing that every equivalent norm on a separable Banach space can be approximated by C^k-smooth norms, provided the space admits an equivalent C^k-smooth norm [431999]. This optimal result solves a 20 years old problem stated in the Deville-Godefroy-Zizler monograph.

The book [358155] coauthored by team members M. Fabian, P. Hájek and V. Zizler consisting of more than 800 pages contains a rather complete background for Banach space theory and its applications. Just for getting a taste, let us mention some from 17 chapters: Schauder bases, Finite-dimensional spaces, Optimization, Higher order smoothness, Dentability and differentiability, Weakly compactly generated spaces, Tensor products. The monograph is based on the previous successful book Functional Analysis and Infinite-dimensional Geometry of the same authors (together with J. Pelant) which was published in Springer Verlag in 2001. However, the topics of the present book are essentially extended (the size of the book is almost doubled).

The monograph [364831] is the first book that applies recent developments and classical results in descriptive topology to the study of classes of infinite-dimensional topological vector spaces that appear in functional analysis. It contains both classical and very recent results. A significant part of the monograph was written by W. Kubiś and it is devoted to topological methods in Banach space theory and a survey of important classes of compact spaces. Many of the results therein are due to W. Kubiś, some of them appeared in the book for this first time.

2.2. The most important results of the team within the Banach space theory

Contributions to this area were done by team members M. Fabian, P. Hájek, W. Kubiś, and V. Zizler. The results obtained fall into several areas of the so broad Banach space theory, in particular, to the following topics: Szlenk indices (Hájek), Lipschitz free spaces (Hájek), renormings (Fabian, Hájek, Zizler), tensor products (Hájek), networks in Banach spaces (Fabian, Kubiś, Zizler), separable complementation (Kubiś), variational analysis in Banach spaces (Fabian). More detailed description of the obtained results follows.

In the paper [353270], some facts about Gul'ko, descriptive, Gruenhage, and fragmentable compact spaces are collected. In several instances, new, more direct, proofs are presented. It is then demonstrated how the proved statements reflect the geometrical structure of corresponding Banach spaces C(K). In particular, it is shown how a simple transfer of Day's norm leads to recent renorming results of the duals to C(K), originally proved by M. Raja and R. Smith.

It was proved that in the space $L_1(\mu)$, and more generally, in the corresponding Lebesgue Bochner space, a carefully selected Orlicz function produces an equivalent norm, of Luxemburg type, having surprisingly nice rotundity and smoothness properties. As a side effect this extends older results of H. Rosenthal. This interesting result was achieved by M. Fabian in collaboration with S. Lajara [380501].

The classical theorems of Lyusternik-Graves on local openness and regularity of mappings was significantly extended, when approximating a mapping in question by a bunch of linear surjections. Using a separable reduction, M. Fabian and R. Cibulka were able to avoid the use of the deep Brower's fixed point theorem [395288].

Several statements on Fréchet subdifferentials, useful in non-linear and non-smooth analysis, were tackled via elegant algebraic approach, and thus the already existed separable reductions were drastically simplified in a joint result of M. Fabian and A. Ioffe [422304].

Furthermore, an optimal estimate of the dependence of the Szlenk index of a Banach space on the dentability index was obtained by P. Hájek and Th. Schlumprecht [427768]. Previous results in this direction were based on the Kunen-Martin theorem in descriptive set theory, and they were only existential.

Schauder bases have been constructed by P. Hájek and E. Pernecká [427496]. This is an improvement of the work of several authors, including Godefroy and Kalton, concerning approximation properties in this kind of spaces.

A nearly optimal result on the approximations by means of polynomial algebras was obtained by S. D'Alessandro and P. Hájek [422561].

Constructions of LUR and simultaneously C^1-smooth renormings of certain Banach spaces, including the WCG spaces was presented by P. Hájek and A. Procházka [425968] solving a problem from the Deville-Godefroy-Zizler monograph.

Additionally, more precise version of the Grothendieck duality result which is widely used throughout the theory of tensor products was proved by P. Hájek and R. Smith, thus solving a problem from the monograph of Defant-Floret [386077].

Another direction of research in Banach space theory was studying purely topological properties of weak topologies, in particular the existence of special networks. One of the results, obtained by W. Kubiś with coauthors [arXiv:1412.1748] says that the Banach space $l_\infty(R)$ with the weak topology is a so-called *aleph-space.* This provides a very natural and concrete example distinguishing certain topological properties. Other results of W. Kubiś include a study of Banach spaces of continuous functions over compact spaces admitting a continuous semilattice operation. For a large class of such compact spaces, their continuous function spaces have special networks, related to certain renorming properties [422355].

Finally, Banach spaces with many projections onto separable subspaces were studied by W. Kubiś. In particular, he obtained, jointly with O. Kalenda [364830], full characterizations of separable complementation properties of C(K) spaces, where K is a compact linearly ordered space endowed with the interval topology. Similar results, for different compact spaces K, were obtained by Kubiś with coauthors [391353].

3. Universal homogeneous structures

The theory of Fraïssé limits deals with classes of "small" structures satisfying the amalgamation property, and the key point is the existence of a unique special "big" object (called the Fraïssé limit) encoding all the information about the class. Fraïssé theory is an important well-established tool among model theorists, where one deals with a fixed class of structures and looks at possible embeddings between them. It was already shown by Droste and Göbel (1992), who actually started category-theoretic foundations of Fraïssé limits, that even restricting the class of embeddings to a smaller category leads to new applications.

W. Kubiś successfully developed a category-theoretic framework for the Fraïssé theory of universal homogeneous structures in [430292]. The key feature of Fraïssé construction is extracted and the notion of a Fraïssé sequence, that is, a sequence of small structures leading to the Fraïssé limit, defined. This idea is very general and shows that category-theoretic framework is the right one for studying Fraïssé limits in broader contexts. The work of W. Kubiœ contains a thorough investigation of the concept of the Fraïssé limit from the category-theoretic perspective, showing several new applications. It is worth mentioning that even as a by-product of the general theory, Kubiś obtains the existence (assuming the continuum hypothesis) of a unique isometrically universal Banach space of density aleph one, that is at the same time homogeneous with respect to isometries between separable subspaces. It turns out that this space has not been discovered earlier, yet it plays an important role in the study of extension properties of Banach spaces (and was named *Kubis space* in the papers by A. Aviles, C. Brech [AB, 2011] and by A. Aviles, F. Sanchez, J. Castillo, M. Gonzales, Y. Moreno [ASCGM, 2011]. Actually, the above mentioned article of W. Kubiś contains much more delicate and interesting examples of Banach spaces that come out as Fraïssé limits, for example, a universal Banach space with a monotone Schauder basis of length ω_1.

The work described above has several continuations. The most important one deals with approximate (also called continuous) Fraïssé limits, where the category is enriched over metric spaces. It was inspired by a short note [395785], where elementary proof of the uniqueness of the Gurariĭ space, the unique isometrically universal separable Banach space with a rich group of linear isometries, was found. The preprint [arXiv:1210.6506] contains both the theory and new examples, explaining some known objects as general Fraïssé limits. It was proved that several well-known structures in functional analysis and topology can be explained within the framework of continuous Fraïssé limits.

Within this topic, the works coauthored by W. Kubiś [430294] and [443126] contain new constructions of continuous Fraïssé limits in categories of quasi-Banach spaces.

Yet another work of Kubiś provides category-theoretic characterizations of retracts of Fraïssé limits, significantly extending previous results coming from model theory [443127].

4. Function spaces

Function spaces constitute a central part of functional analysis since about 1950's and they have seen a true renaissance ever since the early 1990's. Their indispensable role in solving difficult problems appearing in mathematical physics, PDE's etc., is widely known. It seems that their importance is recently dramatically increasing in connection with new tasks. We have seen at several occasions that a choice of an appropriate function space meant a decisive breakthrough to a problem (to name just one example for all, let us recall the role of the variable Lebesgue space in the investigation of properties of electrorheological fluids).

Among the most important problems within this topic studied by A. Gogatishvili were function spaces defined on metric measure spaces. Using an interpolation approach, he defined the Besov type spaces on a doubling metric measure space supported a Poincaré inequality. Under some assumption it was shown that this definition of a Besov space is equivalent with the definition given by Bourdon and Pajot. Several optimal characterizations of Besov and Triebel–Lizorkin spaces, including a pointwise characterization were established in [339255] and in [393457].

Important properties and applications of basic integral operators of harmonic analysis, including certain class of sublinear operators, generalized Riesz potential and Calderon-Zygmund singular integral operators in general Morrey type spaces were also studied

([346690], [348182]). In particular, A. Gogatishvili has shown that the problem of boundedness of the fractional maximal operator in general local Morrey-type spaces is reduced to the problem of boundedness of the supremal operator in weighted Lebesgue spaces on the cone of non-negative non-decreasing functions. This allowed to obtain sharp sufficient conditions for boundedness for all admissible values of the parameters, which, for a certain range of the parameters wider than known before, coincide with the necessary ones.

Furthermore, A. Gogatishvili studied boundedness of general quasi-linear operators in weighted Lebesgue spaces restricted in the cone of monotone functions. He obtained full solutions of many open problems in this field [395276].

The boundedness of Stein's spherical maximal function in variable Lebesgue spaces was proved and application to the wave equation investigated [393004].

Finally, generalized Lorentz spaces were introduced and studied. In particular, their Köthe dual was determined. As an application of this result it was obtained a necessary and sufficient condition for a Sobolev-type embedding into the space of bounded functions [434068].

5. Complex analysis

Within this topic the highlights among the results of M. Engliš include, first of all, a resolution of the famous conjecture of Ramadanov concerning the logarithmic term of the Szegö kernel. Together with the Bergman kernel, this is a certain function associated to a domain in the complex n-space which is highly relevant for many problems in the analysis of several complex variables and in complex geometry, with applications in mathematical physics, machine learning, and elsewhere. Description of the boundary behaviour of these kernels was a ground-breaking achievement of C. Fefferman, for which he subsequently received the Fields Medal; he proved that both kernels can be expressed as a sum of a smooth term and of a smooth term multiplied by the logarithm of the boundary distance. The conjecture asserted that the logarithmic term is actually always present, except for domains biholomorphic to the ball (at least locally). It was subsequently studied by many well-known authors and proved in various special cases (e.g. by Boichu & Coeure, Inv. Math. 1983; Boutet de Monvel, 1982; C.R. Graham, 1987; Lu and Tian, Duke Math. J. 2004; etc.) The paper of Engliš (jointly with G. Zhang) [357381] came a bit as a surprise, due to giving a counterexample in the form of the disc bundle over a compact Hermitian symmetric space, thus settling a problem which has been open for 29 years, despite considerable efforts of many experts in the area.

Another highlight in the area of symmetric spaces is the joint work of Engliš with H. Upmeier (Marburg) on unified group-theoretic framework for Berezin and Berezin-Toeplitz quantizations on such spaces, which materialized in the form of a series of 5 papers (four articles were published in the evaluation period [386415], [373159], [351125], [434137] - the last one appeared in Trans. Amer. Math. Soc. in 2015).

Apart from symmetric spaces, the second main direction of research concerned analysis and operator theory on strictly pseudoconvex domains. A highly novel result established by M. Engliš was the existence of the analytic continuation of weighted Bergman kernels with respect to the weight parameter, which appeared in the highly prestigeous J. Math. Pures Appl. [357384], while a number of papers discussed the topic of membership of Toeplitz and Hankel operators (and their products) in Dixmier classes in various settings [357382]. The results on analytic continuation is somewhat reminiscent of the classical existence result for the analytic continuation of Riemann's zeta function and likewise settled what was a kind of "folklore conjecture'' among the experts; while the work on Dixmier classes was motivated by the high relevance of the Dixmier trace in noncommutative geometry, on the one hand, contrasted to the fact that even now surprisingly little is known about the Dixmier class membership even for the Hardy space Toeplitz/Hankel operators.

Another application of the techniques developed there was also a partial solution of the Arveson-Douglas conjecture in multivariable operator theory [441811] (jointly with J. Eschmeier, Saarbrücken), which was published in Adv. Math. in 2015. The conjecture concerned the size of commutators of model operators (compressions of the backward shift), which is highly relevant for the structure of algebras generated by them; it goes back to Arveson in 2001 and Douglas in 2003, with numerous contributions by Davidson, Shalitt, Wang, and others. While a number of authors proved various special cases (e.g. the case of two or three operators, or of operators on some special space), the work of Engliš and Eschmeier is the first to deal with a fairly general case and represents basically the first major advance in the problem.

6. Thermomechanics of deformable solids

The theory of deformable solids deals with real-life bodies and as such has numerous practical applications. It forms a basis for modeling in engineering and in numerical mathematics, for example. The research within this topic performed by a team member M. Šilhavý developed into the following three directions: *Masonry bodies, fiber and membrane reinforced bodies,* and *phase transitions*. Roughly speaking, the first two of them are related to civil engineering questions (statics of masonry structures) and the third one to material theory (smart materials, shape memory alloys). All these topics are the subject of the contemporary research.

Masonry bodies: The Fenchel duality theory was applied to examine the limit analysis for no—tension bodies used to model masonry structures, with the surprising result that the static and kinematic theorems generally do not provide the same results [392481]. Further, the class of safe loads has been examined using stresses modeled as measures with divergence measure. The integration of parametric measures was introduced to smear the singularities in measures to obtain square integrable stressfields [360205]. A new existence theorem, based on the hypothesis of the absence of collapse mechanisms, has been proved [434070].

A new approach to fiber and membrane reinforced bodies has been developed treating these inclusions as ideal curves and surfaces bearing a deformation-dependent energy ([393470], [428616]). A direct, invariant formalism for nonlinear shells with application to surface-substrate interactions has been introduced with a view of application to shell inclusions [422121].

Phase transitions: Both the sharp and diffuse interface theories have been extended with a new existence theory based on the condition of interface quasiconvexity ([360209], [348282]). The asymptotics of the effective energy in the Allen-Cahn model with deformation has been established in one spatial dimension [360202]. Maxwell's relation for phase equilibria in isotropic solids has been determined [399540].